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THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

SOME ASPECTS OF SPATIAL ABILITIES

AND

ELEMENTARY SCHOOL MATHEMATICS

by



FRANK RIGGS

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THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled SOME ASPECTS OF SPATIAL ABILITIES AND ELEMENTARY SCHOOL MATHEMATICS submitted by FRANK RIGGS in partial fulfilment of the requirements for the degree of Doctor of Philosophy.





## ABSTRACT

This study was motivated by the increased emphasis being placed on geometry in mathematics programs of elementary schools. The main purpose of this study was to make suggestions for and to provide examples of activities for elementary grade children. Since those activities were to centre around spatial abilities, a secondary purpose of the study was to investigate the development of some aspects of spatial abilities in young children.

One hundred children from playschools, kindergarten, Grade I and Grade II were individually administered four tests on spatial abilities dealing with the areas of Topological, Euclidean, and Projective Space and a test on the Co-ordination of Perspectives. A total of seven items from the tests of Topological and Euclidean Space required children to draw two-dimensional shapes by using pencil and paper. In order to compare the responses represented by pencil and paper with some other mode of representation 95 different children represented those same shapes by using strips of plasticine. It was found that there were no significant differences in the mean scores of the two groups.

The remaining analyses centered around the responses of the 100 children who were administered the tests of Topological, Euclidean and Projective Space and the Co-ordination of Perspectives. No significant differences existed on the variable of sex on any of the tests. Age was significantly related to the various aspects of spatial abilities.

On the test of Topological Space the mean scores on the categories of closure, proximity and separation were not significantly



different. Ordering from a 'figure-eight' configuration was most difficult for younger children.

The categories on the Euclidean Test which required accuracy of lines and angles were significantly more difficult than categories of closure, and presence of angles and lines. The categories dealing with horizontal and vertical transformations were quite difficult for younger children.

Constructing oblique lines, as investigated on the test of Projective Space, was found to be significantly more difficult than constructing straight lines near the edge or across the centre of the base. Older subjects responded with a much greater degree of accuracy than younger subjects.

On the test of Co-ordination of Perspectives two of the four views which the subject was required to imagine contained one less object than the view from the subject's position. The accuracy of responses on those positions was significantly less than on the positions containing the same number of objects as the subject's viewpoint. It was not found, however, that older subjects did significantly better in responding to the most difficult positions.

There was general agreement across the tests of Topological, Euclidean and Projective Space regarding the age at which topological and Euclidean spatial abilities develop in young children. However, the ability to represent straight lines and angles, without attention to length and size, appears to be no more difficult than topological properties.

Several suggestions were made for further research. The final chapter contained suggestions and sample activities for elementary school mathematics.





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## CHAPTER I

### INTRODUCTION AND STATEMENT OF THE PROBLEM

Mathematics in primary and elementary grades traditionally consisted of a division of mathematics termed arithmetic. Geometry and algebra were usually reserved for secondary grades. During the past two decades emphasis has been on the unity of mathematics which has resulted in geometry being included in the mathematics curriculum of the elementary school.

The difficulty in this connection has not been whether to include geometry but rather what geometry should be included in a mathematics program for young children. Most current elementary mathematics programs have adopted the geometry of the secondary school programs and stated the ideas in a simpler way. The result has been that elementary grade children are being taught Euclidean geometry sequenced to correspond not to the cognitive processes of the learner but primarily to the logical development of the discipline of mathematics.

It is doubtful that young children can benefit from such an approach. It is considered as a meaningful and an enjoyable activity for young children to play games with concrete objects but it is doubtful that such young children as are normally found in the elementary grades can benefit greatly from a game involving abstractions in rules, assumptions and generalizations. This is what learning mathematics has, perhaps necessarily, become. Mathematics is a game for mathematicians. But surely, activities in mathematics, as in other disciplines, must be so devised that in addition to having relevance to the discipline they



must also aid in the child's development. If a child has not developed to the point of reasoning abstractly, then activities demanding this kind of thinking may not be very helpful.

A more suitable approach to mathematics in the elementary grades might be based on the 'normal' development of the child. Such an approach to geometry would be based mainly on the child's spatial ability. The Ontario Geometry Project - K-13 (1965) contained a similar idea. This report stated that "one of our aims in attempting to revitalize school geometry is to free it from the restricting aspects of logic and to emphasize its active relationship with art and culture (p. 5)". The Cambridge Report (1967), dealing with teacher education, suggested instruction in spatial relationships and the study of shapes (pp. 8-9). In commenting on the idea of spatial abilities Lovell (1971) writes, "There is little doubt that spatial work has in the past, been neglected in the education of young (p. 69)". Elaborating on this point he states:

Children, however, are unable to grasp abstract structures, so if we want to stress them we must find concrete realizations of such structures in the real world. Spatial relationships are often rich in such structures at varying levels of difficulty (p. 69).

Lovell suggests that the approach to geometry through a point-line-set theory has no place in the elementary school. Copeland (1970) expresses a similar viewpoint. He prefers, as do Dienes (1960) and Lovell (1971), to approach geometry through familiar settings and everyday experiences.

The suggestions for geometry activities expressed by Copeland, Dienes, and Lovell emerge from the research and theory of Jean Piaget





(1960a, 1960b, 1967). Only since the changes in mathematics which resulted in the term 'new mathematics' have Piaget's ideas on the nature of the learner and the nature of learning been studied by mathematicians and mathematics educators. It is felt that Piaget's ideas on spatial abilities have implications for suitable geometry activities for elementary grade children.

## I. PURPOSE OF THE STUDY

The main purpose of this study was to make suggestions for, and provide examples of, geometry activities for elementary grade children. These suggestions were based on the findings from research on child development and especially from the study of the subject's spatial abilities. A secondary purpose, therefore, was to investigate some aspects of the development of spatial abilities in Playschool, Kindergarten, Grade I and Grade II children by studying the subject's responses to tasks in the areas of Topological, Euclidean and Projective Space and a test on the Co-ordination of Perspectives.

Attention was given to the following:

- (a) Differences in two modes of representation of geometrical shapes.
- (b) Relative difficulty of various aspects of spatial abilities in the various areas being tested.
- (c) Order in which spatial abilities develop in children. That is, whether the order is Topological, Euclidean, Projective or some other order.

## II. STATEMENT OF THE PROBLEM

The basic problem was concerned with obtaining information to assist in developing geometry activities in elementary grades. The position taken in this study is that elementary geometry must be con-



cerned with the spatial abilities of children. Spatial abilities were investigated by examining subjects' responses on the test items in the areas of Topological, Euclidean and Projective Space and the Co-ordination of Perspectives.

A number of test items required that pupils draw geometric shapes. In regards to those drawings it is conceded that inaccurate responses may have resulted because of the limitations of the subject's motor integrations. Therefore exercises which did not form a part of any test were given to all subjects to indicate any gross deficiencies in this connection. Research reported by Piaget and Inhelder (1967) and Olson (1970) indicates that for all other subjects drawings which are of a fairly primitive nature offer a good indication of a child's spatial ability.

An advantage of the use of drawings arises from the fact that every drawing must necessarily possess topological and Euclidean properties. This, in conjunction with the presence of topological properties in the test of Projective Space, provided an opportunity to investigate not only the topological aspect of the responses within a single test but also a subject's response relative to topological properties across the three tests of Topological, Euclidean and Projective Space. Suppose, for example, that a subject were successful in regards to topological abilities on all three tests but unsuccessful in the areas of Euclidean and projective space. This finding that the topological aspect is so pronounced would permit more definitive conclusions regarding the spatial abilities of that particular subject.

The order in which spatial abilities develop in children was also investigated. On the basis of a subject's existing spatial abilities



and the order in which spatial abilities develop in young children geometry activities were suggested. Using this approach, geometry for the elementary grades could be sequenced to provide experiences which would aid in the development of spatial abilities. Those abilities could in turn be applied to the appropriate area of structured mathematics.

### III. DEFINITIONS

Closure. A drawing in two-dimensional space is closed if each of its components is formed by a continuous set of points with no intervals.

Proximity. The relative location of components of a model represented in two-dimensional space. It concerns the 'nearness' of one component to the other.

Separation. Components of a model represented in two-dimensional space having not more than one common point.

Topological Space. Space characterized by the attention given to proximity, order, separation and closure of an object or its pictorial representation. Topological ability of a subject is not considered to be influenced by his attention to the notions of rigid shapes, distances, and angles or to mensuration and projective relations.

Euclidean Space. Space characterized by attention to angles, distances, proportions and parallels, and the ability to operate in a co-ordinate system. Euclidean Space presumes the co-ordination of the actual locations relative to which other objects are moved.

Projective Space. Space characterized by the ability to view an object or pattern in relation to a point of view other than that of the observer.



Co-ordination of Perspective. The integration of Topological, Euclidean and Projective spatial abilities.

Pre-Operational Thought. Thought which is not reversible and which has not developed to an extent to overpower the child's perceptual field. Possesses static and passive characteristics. Thought which is not systematized.

Operational Thought. Thought which is characterized by the ability to reformulate, in reverse, the various stages in the solution of a problem. It is not predominated by perceptual influences. More mobile and active than pre-operational thought.

Group 1. One hundred subjects to whom the tests of Topological, Euclidean, and Projective Space and Co-ordination were administered.

Group 2. Ninety-five subjects who represented the shapes in the Topological and Euclidean tests by using plasticine.

Developmental Scale. Scale derived on the basis of category difficulty for each test used in the study.

#### IV. MAJOR HYPOTHESES

The responses were investigated to determine the development of the subject with respect to spatial abilities. On the basis of this investigation suitable geometry activities for elementary grades were suggested.

In the process of investigating the subjects' responses a general analysis will be reported and in addition the following hypotheses will be tested:





1. There is no significant difference between results obtained for boys and girls on the test of
  - (a) Topological Space
  - (b) Euclidean Space
  - (c) Projective Space
  - (d) Co-ordination of Perspectives
2. There is no significant difference between
  - (a) the mean scores on each of the categories of Closure, Separation, and Proximity on the test of Topological Space.
  - (b) the distributions on each division on the developmental scale for the test of Topological Space.
3. There is no significant difference between
  - (a) the mean scores on each of the categories of Closure, Presence of Angles, Presence of Straight Lines, Accuracy of Angles, Accuracy of Straight Lines, on the test of Euclidean Space.
  - (b) the distributions on each division on the developmental scale for the test of Euclidean Space.
4. There is no significant difference between
  - (a) the mean scores on each of the categories of Line Parallel to the Edge, Central Line and Oblique Line on the test of Projective Space.
  - (b) the distributions on each division on the developmental scale for the test of Projective Space.
5. There is no significant difference between
  - (a) the mean scores on each of the four positions on the test of Co-ordination of Perspectives.
  - (b) the distributions on each division on the developmental scale for the test of Co-ordination of Perspectives.

## V. LIMITATIONS

Since some of the test items used in the study required subjects to respond by drawing it would have been helpful, in the interpretation of the statistical analysis, if the extent of motor ability could have been distinguished. The decision to require subjects to



draw the models was made after a number of considerations. Piaget and Inhelder (1967), in an experiment involving a similar test, report a "fairly close correspondence between the stages reached in figure construction irrespective of whether matchsticks or drawings are employed (p. 78)". In another experiment by Piaget (1967) the subjects were required to choose from a number of cards the one which was a duplication of the model card. Piaget reports dissatisfaction with this procedure since he found it difficult to assemble a set of cards which distinguished between very easy and too difficult choices. To help overcome this problem the shapes in the tests of Topological and Euclidean Space were represented by drawing with pencil and by forming the same shapes by using plasticine. The responses on the two media were compared.

The sample size was insufficient to permit categories on the basis of intelligence scores. Hence, any conclusions are generalized at most to urban children of ages four to eight years of a middle I.Q. range.

There may have been much variety in the past experiences of the sample included in the study which may have influenced the results. Insufficient information was accessible to consider this as a variable in the analysis.

The findings were interpreted on the basis of the theoretical background. However, because of the complex nature of child development and the uncertainty attached to the present theories of cognitive development some of the conclusions are not very definitive.



## VI. SIGNIFICANCE OF THE STUDY

Piaget and Inhelder (1967) report a detailed and comprehensive study of the child's concept of space. Other investigations (Page, 1959; Lovell, 1959; Peel, 1960; Laurendeau and Pinard, 1970) have replicated some of Piaget's experiments with a view to investigating further the spatial abilities of young children in relation to the more global aspect of child development.

This study is considered to derive significance from the fact that little research has been carried out in the area of children's spatial abilities as they relate to elementary geometry. It is considered desirable to have more knowledge in this area to assist in suggesting suitable kinds of geometry for elementary grades.

Long range significance of this and similar studies is that such studies should provide information to permit developing a mathematics curriculum which is to a greater extent congruent with the development of children. Further, such information when implemented at the instructional level should assist children to derive increased benefits from their experiences in mathematics.

## VII. THE EXPERIMENTAL SETTING

The following is an overview of the experimental design. A more detailed account is reported in Chapter III.

The sample of one hundred and ninety-five subjects, ranging in age from 47 to 103 months, was randomly chosen from a population of urban children attending schools administered by the Edmonton Public School Board, Edmonton, Alberta. For testing purposes the sample was randomly divided into two groups of 100 subjects and 95 subjects





respectively to permit comparison on different media used in representing shapes.

Tests of Topological Space, Euclidean Space, Projective Space and a test of Co-ordination of Perspectives were constructed by the investigator for use in the present study. The theoretical framework and some test items were derived from the work of Piaget and Inhelder (1967). These tests were administered individually, after a short training session, to each of the group of 100 subjects. These subjects were required to represent the shapes which constituted items 1, 2, and 3 of the Topological Test and items 1, 2, 3, and 4 of the Euclidean Test by using pencil and paper. They responded to the remaining items by using the response apparatus provided. This group of subjects will be referred to as Group 1. The 95 subjects who constituted the second group were individually administered a single test made up of the same items as listed above. Those subjects represented the shapes by using plasticine. In future reference this group will be referred to as Group 2. The shapes represented by the subjects in Group 1, who used pencil and paper and the shapes represented by Group 2, who used plasticine, were identical except for the mode of representation. The only purpose of dividing the sample into two groups was to permit at least a tentative conclusion regarding the ability of young children to represent shapes by drawing as compared to representation by using plasticine.

An analysis was carried out to investigate the possible differences in the modes as related to the representation of geometrical shapes. The main analysis consisted of an examination of the responses in Group 1 who were administered all tests (Appendices A, B, C, and D).



The responses were studied regarding their relationship with age and sex, the relative item difficulty, and the developmental characteristics of various aspects of spatial abilities.

#### VIII. OUTLINE OF THE REPORT

A review of selected relevant literature will be presented in Chapter II. Chapter III contains a detailed account of experimental design, testing procedures, and the research procedures used to test the hypotheses. An evaluation of the testing instruments and the results of testing the hypotheses is contained in Chapter IV. Chapter V includes a summary and discussion of the findings and suggestions for further research in elementary mathematics. The final chapter, Chapter VI, includes some suggestions for and examples of activities for elementary grade geometry based on the findings of the study.



## CHAPTER II

### REVIEW OF RELATED LITERATURE

The primary purpose of this chapter is to review briefly Piaget's work and the related research on the development of spatial abilities of young children. Undoubtedly Piaget and his associates at Geneva have provided the foundation and the impetus for much of the work that has been done in the field of the development of spatial abilities. Most research in this area has been carried out by psychologists and hence have been generally concerned with only psychological implications (Elkind, 1964; Olson, 1969; Smedslund, 1961a; Page, 1959; Kersnher, 1971). More recently people who have expressed interest in mathematics education (Lovell, 1971; Adler, 1968; Copeland, 1970; Almy, 1966) have studied the results of investigations into children's spatial ability from a viewpoint of implications for mathematics curriculum in the elementary grades. This research is also briefly reviewed in this chapter.

In addition, Piaget's position on the acceleration of learning will be presented and research dealing with this aspect of learning will be discussed.

#### I. THEORETICAL BACKGROUND AND RELATED RESEARCH

##### Piaget and Spatial Abilities

Piaget's definition of knowledge indicates that to know an object is to construct or reconstruct it, "to transform it and to understand the process of this transformation (Piaget, 1967, p. 8)". There



is, then, a kind of knowledge related to the actions by which the subject transforms the object. It refers to an active aspect of the internal structure through which reality is assimilated. It is termed operative knowledge. A kind of knowledge which focuses on the static, figural aspect of a given situation is termed figurative knowledge. It is related to the direct or pictorial perception of the successive states. It does not involve operational actions. The operative aspect of knowledge is dependent upon intelligence, the figurative aspect is dependent upon perception.

This distinction of a dual system of knowledge also has application to Piaget's claim regarding the development of spatial abilities in young children. "... the evolution of spatial relations proceeds at two different levels. It is a process which takes place at the perceptual level and at the level of thought and imagination (Piaget and Inhelder, 1967, p. 3)".

Piaget states that together with perception and motor activities a child's sensori-motor space begins to evolve from birth. This space continues to develop by being superimposed upon various pre-existing spaces. When this total spatial organization results in new mental structures made possible by speech and symbolic images the child's spatial abilities undergo a massive change marking the beginning of representational or imaginative space. At this stage in the child's development, Piaget claims, there is no smooth, progressive continuation of the abilities which developed in the sensori-motor stage but rather there is redevelopment of those abilities in terms of the mental structures which the child now possesses.





This claim by Piaget seems to form the basis of his work on the child's conception of space. It is the investigation of representational space which is of interest in this study.

Piaget describes sensori-motor space as a space which is practical and organized for that particular stage of behaviour. The absence of the use of symbols prevents the subject from being mentally able to construct it. This space consists of reflexes and the acquisition of primary habits. The most elementary spatial ability which can be grasped by perception is proximity or the 'nearness' of elements in the perceptual field. The relationships of separation, order, enclosure, and continuity also develop early in the sensori-motor stage. Another stage of development is marked by co-ordination of vision and grasping. This co-ordination permits the handling of objects and the permanence of objects and the constancy of shape and size is realized by the child. Later, around the beginning of the second year, relationships of objects are explored.

Lastly,...,with the appearance of the rapid internalized co-ordinations which denote the full intelligent act, there comes into being for the first time, the mental image.... The symbolic function thus evolved facilitates the acquisition of language or a system of collective signs. Hence from being purely perceptual, space has become partly representational... (Piaget and Inhelder, 1967, p. 12).

Spatial representation is not to be confused with spatial actions or mere imagination. Spatial representation is an internalized action.

Spatial concepts can only effectively predict those results (or actions or objects) by becoming active themselves, by operating on physical objects, and not simply evoking memory images of them. To arrange objects mentally is not merely to imagine a series of things



already set in order, nor even to imagine the action of arranging them. It means arranging the series, just as positively and actively as if the action were physical, but performing the action internally on symbolic objects (Piaget and Inhelder, 1967, p. 454).

It is during the development period from sensori-motor to representational space that the child progresses from a somewhat passive relationship with a perceptually dominated field to a more active role controlled by thought and reversible activities. It is not to be assumed, however, that this development is a rapid or a sudden occurrence. Piaget notes that the spatial structure of a child at the sensori-motor level may attend to various details of a topological and Euclidean nature. He maintains, however, that these abilities are not automatically superimposed on the child's mental structure at the time of the appearance of speech and symbolic images; that is, at the beginning of representational space. Piaget and Inhelder (1967) writes:

Although representational space benefits from and has its imagery enriched by forms already developed by perception, it has nevertheless to reconstruct on its own plane, and in the same order of succession the elementary spatial relationships, first topological, then Euclidean and projective (p. 42).

Piaget believes that elementary textbooks writers need to be cognizant of this aspect of spatial development. It is not correct, Piaget states, that because a child can recognize a straight line, a square, a triangle, or other geometric shapes that he, therefore, has the mental structures necessary to imagine or to represent those shapes. Nor can it be assumed that if a child has the ability to represent one aspect of space that he has 'arrived' at that particular stage of development. Research (Piaget and Inhelder, 1967; Laurendeau and Pinard, 1970) indicates that a particular stage of development in which



a child is classified depends on the nature of the total spatial field which the child is investigating. This slow development, according to Piaget, covers the period of the child's life from approximately the age of two to seven years. It culminates in operational space.

Piaget analysed the spatial abilities of young children by investigating three types of relationships based on contemporary geometry. One type was a study of topological relations involving enclosure, proximity, separation and order. In this study Piaget's subjects ranged in age from just over two years in an investigation of haptic perception to nearly twelve years of age in a test designed to provide information on the child's idea of points and continuity. Piaget investigated the aspects of topological space by requiring subjects to draw geometric shapes, by using a number of articles of clothing on a line which the subject was required to duplicate, by studying the child's ability to tie knots, and his idea of a smallest possible square and a shortest possible line.

The second relationship studied by Piaget concerned metric relations or Euclidean space based mainly on distance and shape. This was investigated by requiring children to draw geometric shapes or to form the shapes by using matchsticks, by investigating children's abilities to operate in a co-ordinate system by exposing them to affine transformations on the rhombus, and by requiring the subjects to predict the shape a contained liquid would assume when the container was tilted. Added to this last test was a model of a boat with a mast and both tests were used to test vertical and horizontal concepts.

The third relationship studied by Piaget was projective space in which the concept of the straight line served as the basis of spatial





relationships. Piaget used a number of tests in studying this relationship since in addition to the projective relationship itself he was interested in the development of the projective ability of young children relative to the development of the ability to operate in Euclidean space. To study this latter concern Piaget questioned his subjects on aspects of location on a plan of a model village. To investigate projective abilities and the co-ordination of perspectives Piaget used a test of the projective straight line, projection of shadows, geometric sectioning, and the rotation and development of surfaces.

The responses to the tests were not judged by Piaget to be right or wrong but were rather considered as an indication of the development of the subjects' spatial abilities. Piaget's findings indicated that there is a primacy of topological abilities (around 6-7 years) over Euclidean and projective abilities in young children; the latter two developing simultaneously (around 7-8 years).

#### Other Related Research on Spatial Abilities

The most comprehensive and detailed replication of Piaget and Inhelder's experiments on the spatial abilities of young children has been done by M. Laurendeau and A. Pinard (1970). The five tests which they used were almost identical to those devised by Piaget. They were concerned with stereognostic recognition of objects and forms, construction of a projective straight line, localization of topological positions, concepts of left and right, and co-ordination of perspectives. The same tests were administered to 50 children at each age level between the ages of two and twelve years, distributed at six-month intervals up to the age of five years and at twelve-month intervals



thereafter. In the introduction to their book Piaget credits Laurendeau and Pinard with having a "complete understanding of the hypotheses under consideration, and a continually alert critical sense which prevents them from adopting these hypotheses before having examined all possible objections and tested every conceivable interpretation (Introduction, p. 1)".

Laurendeau and Pinard found no significant differences due to sex. They share somewhat Lovell's (1959) challenges of the genetic primacy of topological relations. They appear reasonably satisfied that there seems in such cases to be a weak *décalage* in operations but "which in all cases gives priority to the topological relations (p. 70)" over curvilinear-rectilinear relations. They hesitate to draw as sharp a distinction as Piaget and Inhelder (1967). Also from their data they are often unable to distinguish those operations which may be of a Euclidean nature from those of a projective nature. Despite the very detailed and careful design and analyses of Laurendeau and Pinard's study, the authors are very cautious in the interpretation of their findings. Generally, with the exception of the point just mentioned, they agree with the findings of Piaget and Inhelder (1967).

However, in the conclusion of their book they remark that because of the uncertainty concerning horizontal *décalages* it was difficult to connect the findings of the experiment to Piaget's theory of intellectual development.

...the only possible conclusion is that the progressive structuring of spatial concepts occurs in a constant order of succession....The importance of a certain form of regularity is that it testifies to the existence of a certain form of organization of intellectual operations (Laurendeau and Pinard, 1970, p. 423).



Page's (1959) study was an attempt to verify the order of succession of stages in spatial development which Piaget established. He used a test of Haptic Perception similar to Piaget and Inhelder's (1967) test. The sample consisted of sixty children ranging in age from two years ten months to seven years nine months. Page's conclusions confirm the primitive character of topological relations and the "apparent fact that simple Euclidean forms are not basic, almost instinctive, patterns, but are slowly built up from more primitive beginnings (Page, 1969, p. 123)". Although Page's findings agree in essential details with those of Piaget and Inhelder he may have been too conclusive in the interpretations from the results on only one test.

Some researchers have challenged Piaget's report on the primacy of topological spatial abilities over Euclidean and projective abilities. Most notable have been Lovell (1959) and P.C. Dodwell (1963).

Lovell's (1959) experiment involved 150 children between the ages of two years eleven months and five years eight months, with a mean age of four years five months and a standard deviation of six months. Twenty-five second year college students were trained as investigators. All the answer sheets were marked by Lovell. The testing included a replication of Piaget's work on Haptic Perception, Treatment of Elementary Spatial Relationships in Drawings, Linear and Circular Order Study of Knots, the Projective Straight Line and the Co-ordinates of Perspectives.

Lovell interpreted his data on the Haptic Perception Test as indicating little evidence that topological properties enable children to identify certain shapes more easily than others. On the next three



experiments the results generally agreed with those of Piaget. In the Projective Straight Line experiment Lovell reports that the subjects were much more advanced than those reported by Piaget. Lovell's final conclusion is that further experiments are necessary before it can be concluded that the child's conception of space begins with topological concepts which are transformed concurrently into concepts of projective and Euclidean space.

Lovell's choice of analysis in his experiment on the projective straight line may have contributed to the disagreement with Piaget's findings. Lovell paid little attention to qualitative analysis of the individual protocols, but rather concentrated on the subject being capable or not; a single dimension. A more detailed analysis of the subjects rather than the problems may have indicated some important findings on the different levels of spatial representation. In this connection Laurendeau and Pinard (1970) write, "Unless it is followed by an analysis by subjects, the analysis by problems cannot unequivocally determine the real significance of certain behaviours (p. 165)".

Further replication of Piaget and Inhelder's work on the spatial abilities of young children was reported by P.C. Dodwell (1963). In his study Dodwell investigated the nature of the progression of spatial concepts and their relationship with age. The sample, ranging in age from five years one month to eleven years three months, consisted of 194 Canadian children with an I.Q. range of 80 to 136, and were drawn from all socio-economic levels. Dodwell's tests were similar to those used by Piaget and Inhelder (1967) and Lovell (1959).

Dodwell interpreted his results as generally supporting Piaget's assertions concerning the development of spatial concepts.





However, Dodwell stated "it was not possible to characterize the great majority of the children studied as being in any one of the particular stages of spatial concept development described by Piaget (p. 161)". Dodwell's classification procedure placed most of his subjects in a category of 'mixed' behaviour. It appears that this may be due, at least in part, to the nature of Dodwell's tests as many of the test items had varying degrees of complexity, requiring responses ranging from intuitive regulation to genuine operational procedures. This being the case, it is surprising that Dodwell did not see fit to discuss horizontal *décalages* as being manifested in the protocols. It is also worth noting that since Dodwell was interested in discovering a purely topological level of functioning in his subjects, his sample should have included those who would be still in the process of constructing their first spatial concepts. Research (Lovell, 1959; Laurendeau and Pinard, 1970; Piaget and Inhelder, 1967; Peel, 1959; Page, 1959) indicates that subjects younger than five years would have been necessary.

Research by Shantz (1971) and Williford (1970) report findings regarding age differences which are inconsistent with most studies on spatial abilities. Both reported no significant differences between age groups on the criterion measure. Shantz's investigation involved the prediction of object location on a test of the co-ordination of perspective. The testing situation appeared highly contrived and this may have created some of the *décalage* for some subjects. In the Williford study the subjects were to respond to test items following a teaching unit on the concepts of rigid motion and congruence. Insufficient information is given in the article concerning the teach-



ing unit to permit any critique on the experiment.

Investigations into specific aspects of spatial orientation in the Euclidean plane was reported by Beilin (1966) and Kershner (1971). Beilin's sample consisted of 180 subjects with a mean age of seven years six months. His investigation dealt with the representation of water levels in jars tilted at various angles. His findings indicated that perceptual confirmation training was superior to verbal instruction and that water level conceptualization in jars tilted at oblique angles was more difficult than in jars placed horizontally or vertically. Kershner's finding indicated that horizontally oriented figures were no more difficult to deal with on a representational level than vertically oriented figures.

Most research on the spatial abilities of children has centered around Piaget and Inhelder's (1967) comprehensive work on the child's conception of space. Generally the findings have been in agreement with those of Piaget and Inhelder that topological spatial abilities of young children develop earlier (age range approximately from six to seven years) than Euclidean and projective abilities which develop simultaneously (age range approximately from seven to eight years).

Lovell (1959) and Dodwell (1963) challenged the viewpoint of many researchers that topological spatial abilities develop earlier in young children than Euclidean and projective abilities. For Piaget, the curvilinear-rectilinear distinction normally presupposes the previous mastery of topological relations. Laurendeau and Pinard (1970) express some reservation in coming to this conclusion. Lovell (1959) disagrees with this finding of Piaget.



A number of people have expressed their opinions on implications of Piaget's work for elementary mathematics. Some of these opinions have been based on both Piaget's work and research carried out by individuals themselves. In other cases writers have reviewed Piaget's work and formulated opinions without much replication of Piagetian type experiments.

Copeland (1970) writes:

The discoveries of Piaget that children's first geometric concepts are topological rather than Euclidean would thus call for the first type of geometric activities in the nursery, kindergarten, and Grade I to be topological in character (p. 164).

He then proceeds to outline activities dealing with the topological aspects of enclosure, proximity, separation, and order. Copeland concludes the chapter "How a Child Begins to Think About Space" with the remark:

If Piaget is correct it is a purely rote exercise to talk of lines as sets of points in the primary grades. In so doing the new math becomes as rote as the old math. Children begin to parrot definitions for rays, line segments, triangles, etc. They are not ready for such concepts. Logically space can be considered in terms of sets of points, so can lines and rays, but not psychologically before eleven to twelve years of age (p. 1969).

"Geometry activities", writes Copeland, "should be selected for children on the basis of their development. Since topological development is first, activities of this sort should come first. Then to help children in abstracting familiar Euclidean shapes, have them begin with familiar objects such as a pencil, fork or ball...(p. 184). The child must form his own mental constructs based on his own physical action of the objects (p. 185)".





Copeland appears to assume that Piaget has been quite accurate in his observations on child development and that his observations have widespread application for young children.

This assumption is also shared by Almy (1966) who suggests a number of implications of Piaget's theory for the education of young children. She views the use of Piaget's experiments as a useful device in assessing the young child's thinking; that is, as a diagnostic device. On the question of sequence in the curriculum she writes, "Piaget's work clearly implies an ordering among conceptual tasks that suggests certain priorities for instruction...Piaget's theory seems to caution those concerned with curriculum construction to give due attention to the scope or breadth of content and activity as well as to the sequence in which it is presented (p. 137)".

Almy also suggests that on the basis of Piaget's theory pupil activity and social interaction should be important considerations in curriculum development for young children.

Although it is agreed by some researchers that Piaget's theory lends assistance to a theory of instruction, this optimism is not shared by Englemann (1971). He questions the cognitive structures as suggested by Piaget and has, in his own investigations, noted some inconsistencies concerning conservation tasks. Englemann regards himself as being quite successful in accelerating learning on a number of high level conservation tasks but at the same time the same subjects were unable to conserve on some supposedly simpler tasks. "So far as I am able to determine", writes Englemann, "the Piagetian theory provides us with nothing more than a set of accurate descriptions about the performance of children at different ages. It does not



provide us with the theory that clearly implies instruction, lack of instruction or evaluation of instruction (p. 126)".

Lovell (1971) suggests that there must be opportunities in everyday situations to help children in elementary grades recognize spatial relations and the appropriate vocabulary. He suggests situations which distinguish concepts of left, right, over, below, and similar prepositions. The use of pegboards and geoboards are emphasized as useful devices to enable the child to explore properties of two-dimensional figures. Paper-folding is suggested as one way to help children develop projective spatial abilities. Generally, Lovell's suggestions emphasize pupil involvement in real life situations and deemphasize the abstract notions found in many current elementary mathematics programs.

Adler's (1968) suggestions regarding implications of Piaget's work for teachers of mathematics have been made on the basis of a review of the work and not on replication of Piaget's investigations. Adler, in his book, Mathematics and Mental Growth summarizes some of Piaget's work as it relates to child development and on the basis of this summary suggests the following as implications for teachers of mathematics:

1. Since the child's mental growth advances through qualitatively distinct stages, these stages should be taken into account when we plan the curriculum. We should use two criteria for selecting the mathematics experiences a child should have at any given age: (a) they should be experiences that he is ready for, in view of the stage of mental growth that the child has reached; (b) they should help the child to advance to the next stage (p. 55).
2. Before introducing a new concept to the child, test him to be sure that he has all the prerequisites for mastering the concept. If he is not ready for the concept, provide him with the experiences that will help him become ready (p. 55).



3. A child's thinking is more flexible when it is based on reversible operations. For this reason we should teach pairs of inverse operations in arithmetic (p. 58).
4. Mental growth is encouraged by the experience of seeing things from many different points of view...We should use not only the traditional synthetic approach but also the analytic approach via coordinates, the vector approach, and the approach that is based on the use of isometries of the plane (p. 58).
5. Physical action is one of the bases of learning...To develop his concepts of number and space, it is not enough that he look at things. He must also touch things, move them, turn them, put them together, and take them apart (p. 59).
6. Since there is a lag between perception and the formation of a mental image, reinforce the developing mental image with frequent use of perceptual data (p. 59).
7. Piaget claims that topological relations are the first geometric relations that are observed by the child...We should be careful not to overdo the formalization of the study of topological relations in the tenth grade and below. Children like the mathematicians of earlier centuries, will not see the need for trying to prove deductively relations that appear to them to be intuitively obvious (p. 60).

Although the implications for teachers as put forth by Adler and others reflect Piaget's position, the suggestions must not be viewed as a theory of instruction for teachers of elementary mathematics. Piaget is concerned with a theory of cognitive development and a theory of development is not a theory of instruction. Shulman (1967) warns against the danger of 'bootlegging' from psychology and putting it directly into a theory of instruction for mathematics" without the necessary intervening steps of careful, empirical testing and research (p 37)".

Another important consideration arises from the findings of Olson (1969) who suggests that the cultural milieu is a very influential factor in the development of spatial abilities. Olson lists three conclusions regarding his investigations into the effects of the





physical and cultural environment on concept development.

First, the general experiences of the children in the physical world appear to have an effect on this development...Second, the language of the culture is highly related to the conceptual distinctions the children are capable of making...(Third) the formal aspects of the culture, particularly schooling, play a clearly dominant role in the development of these conceptual structures (p. 129).

The suggestion is, therefore, that even if Piaget's conclusions regarding child development are valid for children in Geneva, and even if his findings do have implications for teachers of elementary mathematics, it would not be desirable to apply them, without further research, to a Canadian setting.

The topic under consideration is the teaching of elementary mathematics. Perhaps basic to the above discussion on Piaget's work is whether or not teachers of elementary mathematics ought to place emphasis on the development of the child via mathematical experiences or on the development of the structure of the discipline of mathematics. If the latter, then there is great danger that attempts will be made to teach directly the necessary mathematics concepts before the child's cognitive structures permit. The next section discusses this research devoted to acceleration of learning in young children.

## II. PIAGET AND ACCELERATION OF LEARNING

Piaget views learning as being inconceivable without a theoretically prior interior structure of equilibration. If this statement is accepted what does it mean for educators? Should children, beginning in the elementary grades, begin the study of a discipline through its logical structure? Will such an approach build the necessary structures





to permit functioning of the learning processes? The logical structures of mathematics can be very abstract; is this a desirable place to start? Should children be exposed to the disciplines by involving the capabilities of the child as they occur in their 'normal' order of development? In summary, the questions ask whether the structure of the disciplines should take priority over the developmental structures of the child and can learning be successfully accelerated.

Piaget has firm convictions on the role of learning and development but it appears there has been considerable misinterpretation of his theory. Piaget (1967) stated "Learning is subordinated to development and not vice-versa (p. 332)". In commenting on the work of investigators who have supposedly succeeded in teaching operational structures Piaget asks three questions:

...first, is the learning lasting? What remains two weeks or a month later?...Second, how much generalization is possible? What makes learning interesting is the possibility of "transfer" to a new situation... (and third) what was the operational level of the subject before the experience and what more complex structures has this learning succeeded in achieving? (Piaget, 1967, pp. 332-333).

This does not mean, however, that Piaget believes nothing can be done to expedite the development of operational structures. He expresses his belief in the following:

Investigators...have tried to accelerate the development of logical thinking, and we have available today a considerable body of research on what works and what doesn't work. Most of the research has not worked. It hasn't worked because experimenters have not paid attention to equilibrium theory. Learning a fact by reinforcement does not in and of itself result in mental adaptation. (Stendler, 1967, p. 336).



The idea expressed above is clarified further in Piaget's answer to a question on acceleration of the acquisition of the conservation of number. "Well, you can imagine an analogous structure but in a simpler, more elementary, situation (Piaget, 1964, p. 16)". "In other words, acceleration of learning is possible if the more complex structure is based on simpler structures. That is, when there is a natural relationship and development of structures and not simply an external reinforcement (Piaget, 1964, p. 17)".

Piaget's theory states that attempts at exposing learner's to situations involving more complex structures than at present exist in their repertoire demands careful breakdown and selection of simpler structures which are capable of being assimilated. But these simpler structures can only exist, according to Piaget, because of factors governing the child's 'normal' development -- those of maturation, experience, social transmission, and equilibration. This prompts the question "Why, then, teachers?" Piaget's answer is that "the teacher must provide the instruments which the children can use to decide things by themselves. Children must verify, experimentally in physics, deductively in mathematics. A readymade truth is only a half-truth (Piaget, 1964, p. 5)". Piaget views a teacher as being very necessary in the construction of methods. He writes:

Teaching means creating situations where structures can be discovered; it does not mean transmitting structures which may be assimilated at nothing other than a verbal level (Duckworth, 1967, p. 319).

In reference to the questions at the beginning of this discussion Piaget would obviously favour exposing children to the disciplines by involving the capabilities of the child as they occur in



their 'normal' order of development.

### Review of Selected Research on Acceleration of Learning

The research reported in this section is illustrative of attempts by investigators to measure the effects of deliberate efforts to foster learning in children at an earlier than 'normal' age. Two main problems encountered by research of this type are (i) the uncertainty of the lasting effect of what has been learned and (ii) whether the experience has been purely physical or logico-mathematical for the learner.

Beilin and Franklin (1962) carried out an experiment on conservation of length and area with two main purposes. One was devoted to the question of whether a particular logical development results in the simultaneous ability to solve the related problems of length and measurement. The second purpose, and the one which is of concern in this review, dealt with the limits upon the acquisition of measurement operations associated with age when a deliberate effort is made to foster their acquisition through instruction in the concepts basic to such measurement. Their sample included 27 first grade children with a mean age of six years six months and 33 third grade children with a mean age of eight years eleven months. The mean I.Q. for the first graders was 117.6 with a Standard Deviation of 16.2 and for the third graders a mean of 113.0 and Standard Deviation of 12.5. They followed a test-instruction-transfer test procedure.

The data indicated that the first graders profited primarily from the task training and "a little, though not much, from group instruction. Third graders, on the other hand, profit appreciably more





from measurement concept instruction in that they are able to put to use the generalizations taught them (Beilin and Franklin, 1962, p. 615)". Beilin and Franklin also concluded that in spite of training in the task as well as measurement instructions, no child was able to achieve operational area measurement in the first grade.

In this experiment the age of the subjects obviously was a contributing factor to the positive results. Also, despite the fact that the testing appeared to facilitate learning it is not clear that this learning was of a lasting or of a transitive nature. Those two considerations may detract somewhat from the validity of such a procedure.

Wohwill and Lowe (1962) report the results of an experiment devoted to investigating the development of the conservation of number. They chose their sample of 72 kindergarten children with a mean chronological age of five years ten months so as to be representative of children who would still show lack of number conservation "while yet being old enough to be able potentially to profit from the learning experience; in other words, an interaction between learning and development (was) presumed (p. 160)". Each subject was administered a verbal and non-verbal conservation pretest, followed by a training period, and followed by a verbal and non-verbal conservation post-test.

An analysis of variance revealed no significant differences among training groups with respect to learning non-verbal conservation but there were significant results when the overall groups were considered. However, the control group gained more than two other subgroups which clearly contributed to this significance and prevented the experimenters from attributing beneficial effects to any specific



learning condition.

It is important to consider two features of this investigation -- the sample and the training. The sample was so chosen that the majority of the subjects was quite probably on the borderline of being classified as conservers before the experiment began. The training session was not a reinforcement of the pretest items but was designed to build on the child's previously developed structures of thought. That is, the complex structures of number conservation were broken into simpler structures to aid in this conservation. It is not clear why the results were not of a more positive nature. The experimenters suggest that perhaps the abundance of perceptual cues and the nature of the non-verbal conservation test were contaminating factors. However, Wohwill (1960) did have some success in accelerating the concept of number conservation. Piaget (1964) makes reference to work done by Wohwill, in which he showed that the acquisition of the concept of number can be accelerated through introducing additive operations. That is, there was a natural relationship and development of structures and not simply an external reinforcement.

Jan Smedslund (1961a, 1961b) appeared to have been successful in accelerating the ideas of conservation through learning of a stimulus-response type. Smedslund's (1961a) study involved 135 children with ages ranging from five years to seven years with a mean age of six years two months. On the basis of a pretest 48 non-conservers of weight were selected for the experiment. There were 16 in each of three groups -- Group I was allowed to observe empirical conservation of weight on a balance, Group II practised on additions and



subtractions of material with controls on the scales, Group III took only the pretests and the post-tests.

The data indicated that none of the experimental conditions were adequate to develop operational functioning in the principles of conservation. In Smedslund's (1961b) study 11 children who had acquired conservation of weight by control on a balance and 13 children who had acquired conservation prior to the study were shown examples of apparent non-conservation. The results indicated that extinction was quite easy for those in the first group. Apparently, they had learned only an empirical rule.

"The results are seen as consistent with the hypothesis that a concept of conservation acquired by means of external reinforcement does not have the functional properties of a normally acquired 'logically necessary' concept (Smedslund, 1961b, p. 85)". This statement is in agreement with Piaget and something which Smedslund in his earlier work failed to recognize. In commenting on Smedslund's earlier work Piaget (1964) states:

As far as conservation is concerned, Smedslund succeeded very easily with five and six year old children in getting them to generalize that weight is conserved when the ball is transformed into a different shape. Thus Smedslund very easily achieved the conservation of weight by this sort of external reinforcement. In contrast to this, however, the same method did not succeed in teaching transitivity. Consequently, he was successful in obtaining learning of what I called earlier physical experience, but he was not successful in obtaining a learning in the construction of the logical structure. It cannot be obtained by external reinforcement. The logical structure is reached only through internal equilibration (pp. 15-16).

Lefrancois (1966) reports the results of an experiment in the development of the conservation of mass involving 60 five and six





year old children. Subjects were randomly assigned to one of three experimental conditions: verbalizing, non-verbalizing, and no-training. All subjects were administered pretests, post-tests, and retention tests. From the data Lefrancois concluded that "the concept of conservation of mass can be developed in young children prior to the time when they would have acquired it naturally (p. iv, abstract)". From a Piagetian viewpoint there are two important features that may have contributed to the positive results of this experiment -- the age of the sample and the training session. The subjects were of such an age that many might have been on the brink of being conservers of mass anyway. The training session was designed "to lead logically to the operations ... which are manifested in correct conservation responses (p. iii, abstract)". That is, there was effort devoted to simplifying the complex structures. Perhaps detracting somewhat from the apparent positive outcomes is an observation which Piaget made on Smedslund's earlier work. There is no indication from the reported results that the 'learning' that did take place possessed any transitive characteristics. The post-test and retention test concerned the same situation as did the pretest and training session.

One purpose of the research reported by Towler (1967) was to investigate the effects of training on the acquisition of the concepts of continuous quantities. The sample consisted of 120 grade one subjects. His experimental procedure involved a pretest, training session, post-test and post-post-test or retention test. Towler reports very positive results. The training procedures resulted in helping a significant number of non-conservers and partial conservers acquire an understanding of the conservation principle, and what is more





important, with the exception of one subject, all the members were able to transfer their knowledge to a new situation despite an extinction suggestion by the experimenter. It will be noted that Smedslund (1961b) did not obtain the same results. It is difficult to determine whether or not Towler's extinction procedures were as 'conflict' producing as those used by Smedslund.

Many of the findings of research in the acceleration of learning have not been of a strong positive nature in developmental theory. The findings are nevertheless important as far as implications for education are concerned. The research in this area indicates that guidance and structured learning experiences can be beneficial if they are attempted at a time when the conflict produced by those experiences is capable of being assimilated and accommodated by the subjects concerned. It is apparent from the research reviewed that the age of the sample has almost invariably contributed to the obtained results. However, before the efficacy of attempts of accelerating learning can be accurately assessed it seems necessary to be aware of the kinds of learning desired in a situation and the importance of transfer of that which is learned.

#### Implications Arising From Related Research

A very fundamental issue concerns the acceleration of learning, or learning at an earlier than 'normal' age. An important reason for including mathematics in the school curriculum is to ensure that children gain an understanding of mathematical concepts and are able to apply these concepts to 'real world' problems. If it is true that any child can be taught any mathematical concept at any time, then it seems quite



unnecessary to be concerned with an investigation into the 'normal' development of spatial abilities of children. An obvious procedure would be to begin mathematics teaching with the fundamental mathematical concepts and in a logical and somewhat abstract manner proceed to the highest level of mathematics understanding which the learner is able to attain during his stay in an academic institution. On the basis of the literature reviewed on the acceleration of learning such a procedure does not seem justified. It was decided therefore to be necessary and meaningful to investigate the spatial abilities of young children as a basis for suitable geometry topics for elementary grades.

There are three important issues relating to research on the spatial abilities of young children. One is the order of development of those abilities. Piaget (1967) appears more convinced of this order than Lovell (1959), Dodwell (1963) and Laurendeau and Pinard (1970). Inherent in tests of spatial abilities, as in all Piaget type tests, are the possibilities of *décalage* created by different materials and instruction, different scoring procedures and cultural differences. Any or all of those factors might affect the order in which spatial abilities appear to develop. As controlling features in the present study the materials and vocabulary were generally familiar to all subjects. The scoring procedures took into consideration the implications of incorrect responses as well as correct responses. Also, no generalizations of results will be made beyond an urban Canadian setting.

A consideration of possible cultural differences is another important issue. Piaget (1964) agrees that different results may be obtained on investigations into child development because of cultural differences, and Olson (1969) has reported extensively and conclusively



on the effect of the cultural environment. Not taking cultural effects into consideration may have serious consequences on any implications which studies pertaining to child development may have for the school curriculum.

The third issue is the implications of such studies as spatial abilities for teachers of elementary mathematics. Some writers (Adler, 1968; Copeland, 1970) think it appropriate to list fairly specific implications. Others (Lovell, 1971; Almy, 1966) find many implications for elementary teachers on the basis of Piaget's work and their own investigations. Englemann (1971), on the basis of his work, suggests that Piaget has nothing to offer a theory of instruction in mathematics.

This study investigated some aspects of spatial abilities of young children in an urban Canadian setting. The findings were related to mathematics and formed a basis for mathematics activities for elementary grades. These activities are outline in the last chapter of this report.

The results of the investigation into spatial abilities is part of a theory of development. The activities formed on the basis of those results is a move in the direction of instruction but it is not a theory nor part of a theory of instruction. The activities do not propose alternate routes, nor do they provide for individual learning styles. They have not been tried extensively with children of different abilities in different settings. The position taken is similar to that taken by Hilgard (1966) and Guba and Clark (1965). "It is now commonly recognized", writes Hilgard, "that it is not possible to move from basic science directly to applied without a number of intervening steps, some of which require all the ingenuity





and scientific acumen of basic research itself (p. 573)".

The activities suggested in Chapter VI need to be experimented with, evaluated and modified on the basis of the results to permit pupils to receive optimal benefit from the exercises. Concern must be given to the method of presentation, the classroom organization, pupil recording and teacher evaluation. It is only then that the activities will begin to form part of a theory of instruction in spatial abilities.

### III. SUMMARY

One purpose of this chapter was to review not only the empirical research on spatial abilities and acceleration of learning but also to discuss implications for the teaching of elementary mathematics which various writers suggest arise from Piaget's work in this area. Another purpose was to place the present study in perspective regarding these implications for elementary mathematics.



## CHAPTER III

### DESIGN OF THE STUDY, INSTRUMENTATION AND RESEARCH PROCEDURES

#### I. SAMPLE

The population consisted of the enrollments in the middle I.Q. range (approximately 90-110) in six playschools, and in kindergarten, grade one and grade two classes in six elementary schools in the Edmonton Public School Board System. This comprised approximately one hundred playschool children, one hundred and fifty kindergarten children and three hundred each of grade one and grade two children.

The sample of 195 subjects, 112 boys and 83 girls, was chosen by randomly selecting classes from the above schools. It appeared appropriate to measure some aspects of spatial abilities by requiring subjects to draw two-dimensional shapes by using pencil and paper. In order to investigate the possibility that the co-ordination necessary for representation in this mode made it unsuitable for young children the total sample was divided into two groups. Ninety-five subjects represented the shapes by using strips of plasticine and the remaining 100 subjects represented the same shapes by using pencil and paper. The only purpose of the group of 95 subjects was to permit a comparison on the two modes of representation - pencil and paper as compared with the use of plasticine in representing two-dimensional shapes. The shapes which were represented by these modes are items 1, 2 and 3 on the test of Topological Space (Appendix A) and items 1, 2, 3 and 4 on the test of Euclidean Space (Appendix B). Table 1 shows a break-



down of the two groups with respect to age and sex.

TABLE I  
RANGE, MEAN AND STANDARD DEVIATION OF AGE  
AND SEX DISTRIBUTION

|     |                       | Group 1                                     | Group 2                                    | Total  |
|-----|-----------------------|---------------------------------------------|--------------------------------------------|--------|
|     |                       | (Represented shapes<br>by pencil and paper) | (Represented shapes<br>by using platicine) | Sample |
| Age | Range                 | 50-103                                      | 47-96                                      | 47-103 |
|     | Mean                  | 74.68                                       | 74.56                                      | 74.62  |
|     | Standard<br>Deviation | 12.53                                       | 12.83                                      | 12.66  |
| Sex | Boys                  | 55                                          | 57                                         | 112    |
|     | Girls                 | 45                                          | 38                                         | 83     |

The group of 100 subjects (Group 1) constituted the main part of the sample. In addition to drawing the shapes listed above they were also administered tests of spatial abilities involving aspects of Topological, Euclidean and Projective Space and a test of Co-ordination of Perspectives (Appendices A, B, C, and D).

## II. INSTRUMENTATION

### Test of Topological Space

Purpose: To investigate development of topological abilities in young children.

This test was used to investigate the subject's ability to represent topological space. The required responses demanded the subject's attention to those aspects which characterize topological space - proximity, order, separation, and enclosure. It may be that many sub-



jects, in their responses, considered other relations; for example, metric qualities. It is not suggested, however, that this test was any indicator of the subject's spatial ability in excess of his ability to attend to topological aspects. Since the chief purpose of this test was to investigate representation of topological space a scoring scale has been devised which ignores all aspects of the responses other than those being investigated by this test.

### Test of Euclidean Space

- Purpose: (i) To investigate the extent to which errors in drawing are topological or consisting of a combination of Euclidean or Projective errors.
- (ii) To investigate differences in modes of representation of geometrical figures.

In the test of Euclidean Space attention was focused on two aspects -- Topological and Euclidean Space as represented by drawings of geometric figures (Group 1) and by representing the same figures by pieces of plasticine (Group 2).

A scoring scale has been devised distinguishing between topological successes and successes due to operational (projective or Euclidean) solutions. The same scoring scale was used for both modes of representation.

### Test of Projective Space

- Purpose: To determine the approximate age range in which children become capable of operational solutions in their representation of spatial relationships.

The test of Projective Space, limited in this study to the projective straight line, was used by Piaget and Inhelder (1967) to trace the development from topological to Euclidean and projective representations in the child's concept of space. The required responses





in this test must be distinguished from the ability of the subject to recognize a straight line. This test required a mental representation of a straight line and hence must be representative of the subject's internalized thought and not perceptually dominated. In order to be successful the subject must, therefore, have exhibited ability to operate in a space structured in reference to other objects and himself; that is, space which is of a projective nature. The required responses also demanded that attention be given to the object itself, the straight line. Adequate attention can only result from the subject's ability to operate in a co-ordinate system; that is, space of a Euclidean nature. Hence, the test of Projective Space involved knowledge of projective and Euclidean abilities in order to ensure success. Subjects whose action was governed by perceptual cues, while they may have been successful in constructing straight lines parallel to the edges of the bases used - that is topologically successful - nevertheless may have failed in those areas demanding operational responses.

A developmental scale has been devised to distinguish those subjects who responded on a preoperational level from those who responded on an operational level as measured by the test of Projective Space.

#### Co-ordination of Perspectives Test

Purpose: To investigate the integration of the subject's system of perspectives and his system of co-ordinations.

This test of Co-ordination of Perspectives goes beyond the test of Topological Space which involved the concepts of neighbourhood and beyond the concepts of lines, distances, and angles as investigated



in the test of Euclidean Space. The criterion for success in this test was the successful integration of the properties of Topological, Euclidean, and Projective Space into a unit system. It required the subject to be aware of the Topological ideas of proximity, separation and the order of the cones which represented the mountains. It demanded also that the subject organize the Euclidean plane which involved several objects and an external observer simultaneously.

It required the subject to be able to recognize the function of the projective straight line since success on this test required a system to include the many possible viewpoints.

The scoring scale which was used recognized the implications of errors as well as correct responses. This is reported in a later section.

The tests of spatial abilities and their properties are classified in Table II.

TABLE II  
CLASSIFICATION OF TESTS AND PROPERTIES IN THE  
TOTAL TEST OF SPATIAL ABILITIES

| Test        | Properties                                                                                                                                      | Total Number of Items |
|-------------|-------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------|
| Topological | Closure, Separation, Proximity, Order                                                                                                           | 5                     |
| Euclidean   | Closure, Presence of Angles, Presence of Straight Lines, Accuracy of Angles, Accuracy of Straight Lines, Horizontal and Vertical Representation | 6                     |
| Projective  | Horizontal and Oblique Lines                                                                                                                    | 6                     |
| Perspective | Co-ordination of Perspectives                                                                                                                   | 4                     |



### III. TESTING PROCEDURES

All tests were administered individually by the investigator in a private room in the school. In almost all cases the testing area was not close to routine school activity and, hence, it was very quiet. Any objects which were obvious distractors for the subject were removed from the testing area. The total testing time for each subject in Group 1 was approximately 30 minutes. In consideration of age of the sample this time was divided into two test sittings of approximately 15 minutes each. The total testing time for each subject in Group 2 did not exceed 20 minutes. The appropriate parts of both the Topological and Euclidean Test were administered on one sitting.

#### Topological Test - Group 1

Subjects were seated behind a desk of appropriate height and were first asked to 'draw a man' by using a pencil and a 5" by 8" white card. No model card was provided. This exercise was used to serve two purposes. First, it was hoped that it would enable the subject to relax in the unfamiliar setting, and secondly, it was assumed that the drawing would indicate any gross deficiencies of the subject relevant to the response requirements. It was apparent that no such deficiencies were present in any subject of the sample in this study.

Each subject was then presented with a brightly colored thin board on which were placed three 5" by 8" cards, laid one on the other so that only one card was visible at any one time. Each card contained geometric shapes (Appendix A, items 1, 2, and 3). The subject was also given a supply of paper, approximately 6" by 10", and a thick





lead pencil. The subject was required to draw, on separate sheets of paper, each of the three shapes on the cards. In the event that the subject expressed a desire to have more than one try at a drawing, the previous drawing of that particular shape was kept and any relevant comment of the subject was recorded. The subject was not permitted to move any card containing the shapes. There were frequent comments by the subjects regarding characteristics of the shapes. When one drawing was completed to the subject's satisfaction the top card was removed exposing the next shape to be represented, and the subject's drawing was taken and labelled by the investigator. This procedure was followed for each of the first three items in the test of Topological Space (Appendix A). The test setting is illustrated in Figure I.

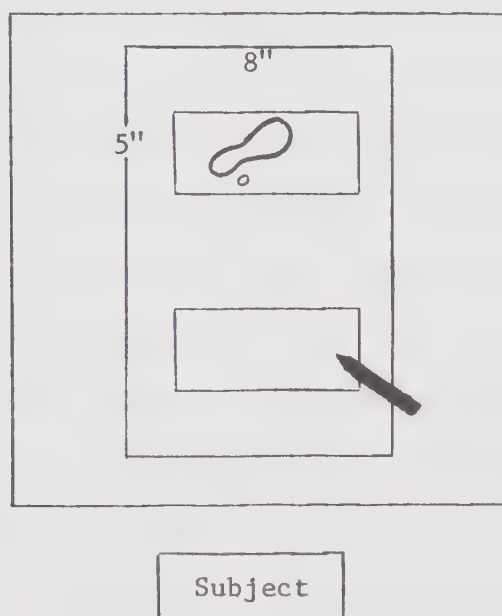


FIGURE I

SETTING FOR REPRESENTATION OF SHAPES BY DRAWING



Test items 4 and 5 of the test of Topological Space have been included to investigate the order notion of Topological Space. Item 4 of the test of Topological Space required that the subject place seven colored beads on a straight rod in the same order of colors as they appeared on a circular shaped wire. First, each subject was asked if he knew what something 'straight' looked like. He was asked to pick a straight object from a collection of various shapes. He was then asked to identify a straight line from three lines drawn on paper. The word 'straight' was emphasized and the subject was asked to repeat it several times in reference to the objects he had chosen. Invariably one of the straight objects was the wire which was to be used in this test item, and it was made certain that 'straight' was used to describe it. The subjects were shown that the ends of circular shape wire containing the beads could be separated and hence could be made straight. The subject's agreement that this was possible was taken as an indication that he understood that part of the instructions.

Next, the subject was asked what he would have to put on the straight rod to make it look exactly like the circular wire if it were straightened. The answer 'beads' was usually given without much hesitation. A tray containing fourteen colored beads - two of each necessary color - was presented to the subject and he was instructed how the beads were to be placed on the straight wire. A check was made to ensure that the subject could match and discriminate the different colors. The subject was then told to start with whatever color he desired and to make the straight rod look exactly like the circular wire if it were straight. He was told that this meant he had to put the right color in the right place. The investigator did



not inform the subject when to stop placing beads on the rod, nor was there any attempt to indicate any error which might have been made in the process.

The procedure for item 5 on the test of Topological Space was similar to that of the previous problem. The response criterion was identical but the stimulus differed in that beads were on a 'figure eight' shaped wire loop instead of a circular shape. As in the previous item it was made certain that the subject understood that the wire formed a continuous shape rather than two independent loops. The same beads were presented as in the previous item.

Scoring. Items 1, 2, and 3 were scored on the criterion of closure, proximity and separation. The subject scored 1 for each correct response. The score on items 4 and 5 were dependent on the number of errors which the subject made in his response. The subject scored 4 on an accurate response and 0 if he committed more than three errors. The total possible score was 17. This scoring was used to arrive at item difficulty, which in turn was used in setting up a general developmental scale as described in Chapter IV.

#### Euclidean Test - Group 1

In this test, items 1-4 were administered in the same way, and in the same sitting, as items 1-3 in the test of Topological Space. That is, the subject represented a total of seven geometric shapes -- four in this test and three in the test of Topological Space -- which were presented to him on 5" by 8" cards in the procedure as described in the previous test. After completion each drawing was collected and labelled by the investigator.



The subject's ability to anticipate and represent locations of moving objects in the Euclidean plane were investigated in items 5 and 6 of this test. In item 5 the subject was shown a transparent beaker containing approximately one inch of colored water. The subject was asked to represent on a sheet of plain white paper, containing a drawing of the beaker, the water as he saw it in the beaker. With an empty beaker the investigator indicated how he would tip the one which contained the water and asked the subject to indicate on a drawing of a tilted beaker how the water would look when it was tipped. Finally, the beaker containing the water was actually tilted and the subject was required to draw the water on the drawing of the tilted beaker as he saw it. That is, the subject made three drawings of the water on three different drawings of the beaker; one drawing showing the beaker upright, and the other two drawings showing the beaker in a tilted position (Appendix B). The procedure for administering item 6 was similar to that used in item 5. The stimulus in this test item was a glass jar containing a plumb line suspended from the centre of the top of the jar and hanging inside in a vertical position so that the line was parallel to the straight sides of the jar and perpendicular to the plane on which the jar rested. The subject was given a drawing of an empty jar and was asked to draw the plumb line as he saw it with the jar in an upright position. An empty jar was tilted slightly and the subject was asked to draw, on another picture of a tilted jar, the plumb line as it would look if the jar were tilted as the empty jar was tilted in the demonstration. Finally, the jar containing the plumb line was tilted and the subject was asked to draw what he now saw on another picture of an empty jar.





As in item 5, each subject drew three drawings of the position of the plumb line - one on each of three separate drawings of the empty jar (Appendix B). The jar containing the plumb line and a sample of the drawings presented to the subjects are illustrated in Figure 2.

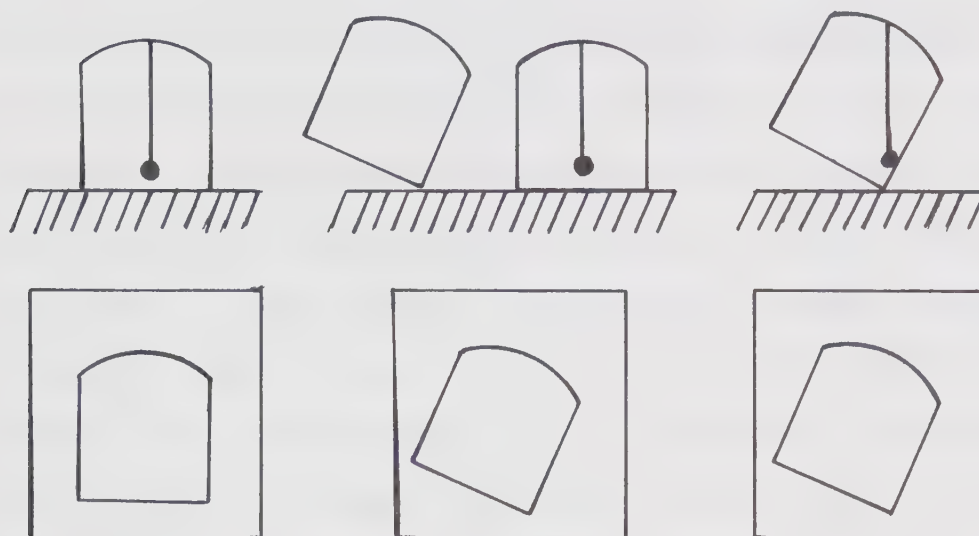


FIGURE 2

#### POSITION OF JAR AT VARIOUS PHASES AND PUPIL PAGES

Scoring. The criterion of closure, presence of angles, presence of straight lines, accuracy of angles and straight lines were used in scoring items 1-4. A drawing was considered to be 'closed' if it was formed by a continuous line or lines. A drawing was considered to have fulfilled the conditions of the presence of angles and straight lines if three of the four or two of the three angles or lines were in evidence. Angles were judged to be accurate if they were within ten degrees of those contained in the geometric shapes which were presented to the subject. Lines were judged to be accurate if the ratio of the lines erred not more than one-fifth of the ratio of the lines in the shapes presented to the subject. The



scoring is described in more detail in Appendix B.

There were three phases of scoring on items 5 and 6. The first phase was the subject's representation of what he saw in the beaker - item 5 - and in the jar - item 6 - when the jars were in an upright position. The next phase required the subject to anticipate the positions of the water and the line if the containers were tilted. The correct response was oblique lines relative to the sides of the containers. Instances where subjects drew both a horizontal and vertical instead of a single oblique were recorded but not scored as being correct. These cases will be discussed in another section of this report. The final phase involved the subjects being able to check what they had anticipated in the second phase. This was scored as in phase two, an oblique line relative to the sides of the container being the criterion for complete success.

In arriving at item difficulty in this test all responses to items 1 and 4 were considered but only responses to phase two of items 5 and 6 since this was the criterion for success on those items. Responses to phases one and three will be discussed in Chapter V. The total possible score on this test was 22.

#### Projective Test - Group 1.

The requirement of the test of Projective Space consists of constructing as straight a line as possible between two fixed points, using about 10 matchsticks each of which is vertically fixed in a small disk of modelling clay. The material consists of a rectangular base (about 25 by 35 cm.) and a circular base (about 35 cm. in diameter), two small houses and about 10 matchsticks to represent lamp



posts or some other familiar object. The test is divided into two parts, each of which contains three problems (Figure 3). In the first part, using the rectangular base, the subject must construct a straight line, by using the matchsticks, between the two houses. First, each of the houses are placed about 2 cm. from the longer side nearer the subject (Problem 1) and then, for Problem 2, at 2 cm. from the centre of each of the smaller sides of the rectangle. In Problem 3, the houses are placed about 30 cm. apart and are positioned as shown in Figure 3. In the second part of the test, the houses are placed on the circular base. The procedure is similar to the first part. In the fourth problem the houses are spaced about 20 cm. apart on a small segment of the arc perpendicular to the child's plane of vision. The houses are placed at about 30 cm. along a diameter still perpendicular to the plane of vision for Problem 5. In Problem 6 the houses are placed about 25 cm. apart on a larger chord of the arc but this time on the oblique in terms of the child's plane of vision. (Appendix C includes test instructions in more detail.)

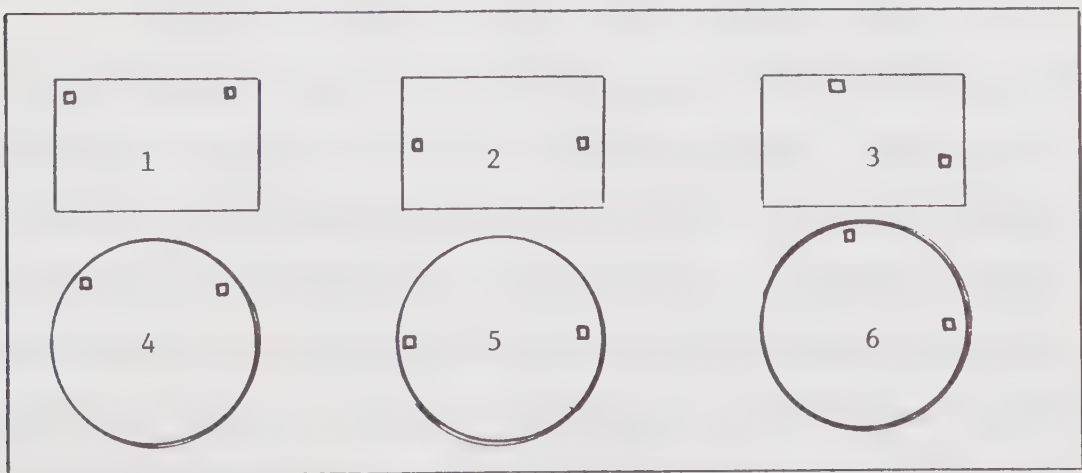


FIGURE 3

## TEST OF PROJECTIVE SPACE





Scoring. The subject was given a score of 1 for each straight line which he constructed between the two designated points on the rectangular and circular bases. The method which was used in constructing the straight line was recorded but did not influence the subject's score. The total possible score on this test was 6. This scoring permitted the investigation of relative item difficulty which was used to establish a developmental scale.

#### Co-ordination of Perspectives - Group 1

The experimental apparatus consisted of a plain white cardboard square measuring 50 cm. on each side and divided by masking tape into four equal quadrants. Three cardboard cones, red (20 cm. base by 10 cm. high), blue (14 cm. by 7 cm.) and green (8 cm. by 4 cm.) were placed on the cardboard square and laid on a table. A set of eight colored sketches which corresponded to pictures of the cones from various viewpoints were placed directly in back of the table layout in a vertical position. The table layout and the sketches were placed as in Figure 4.

The test was administered in the following manner. All of the eight sketches were covered with cardboard flaps which were hinged to the large cardboard sheet containing the sketches. The subject was seated in a position as indicated in Figure 4. He was asked to pretend that the colored cones were mountains. He was next shown a wooden figure of a man (about 6 cm. high) with an object around his neck which resembled a camera. Each subject was asked what a camera was used for and what he thought the little man was going to do with the camera. The investigator then told the subject an imaginary story about the little man taking pictures of the mountains.



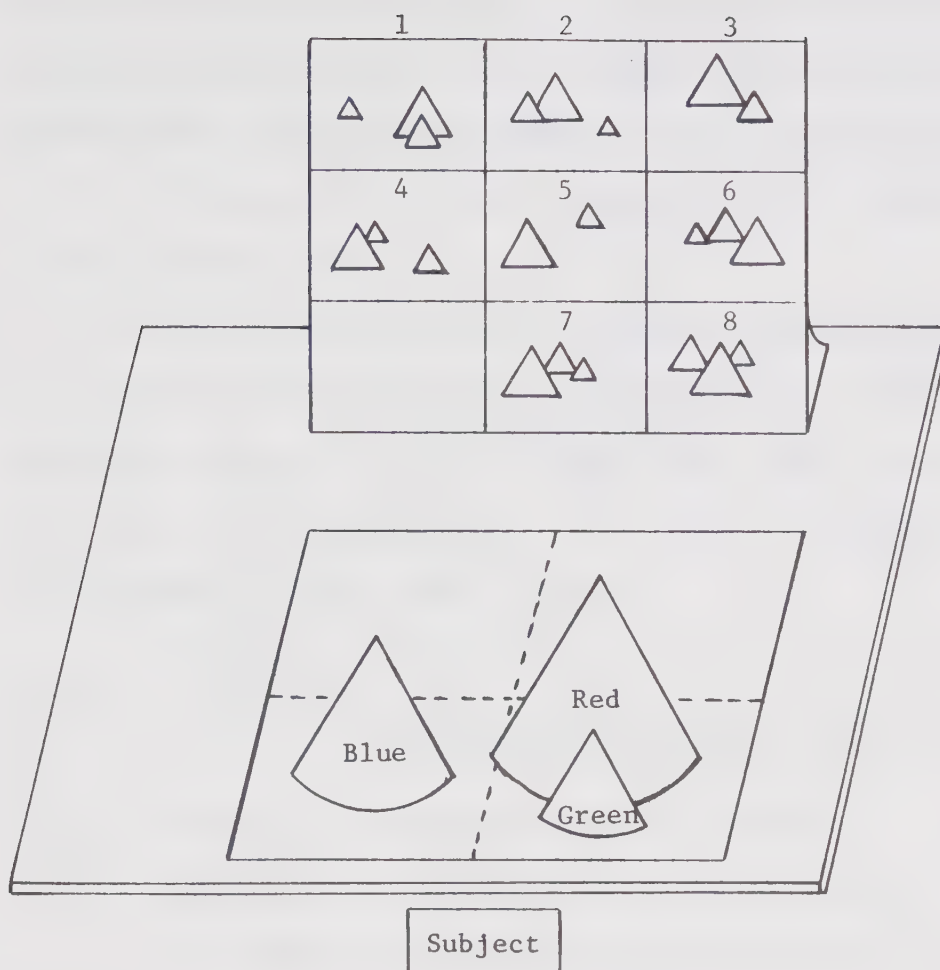


FIGURE 4

## APPARATUS FOR TEST OF CO-ORDINATION OF PERSPECTIVES

A training item was included where the 'photographer' was on the same side and in the same position as the subject. The cardboard flaps were raised to expose sketch 1, the correct view, and sketch 4, an incorrect view. The subject was to choose the picture which he thought represented that taken by the 'photographer'. It was found important to uncover only the pictures from which subjects were to choose and to do so after the man had 'taken his picture'. Otherwise subjects were confused as to exactly which objects they



thought were being photographed. A discussion followed about why the choice was right or wrong. In all cases sufficient information was given so that the subject agreed that sketch number 1 was the correct picture. The selection of a picture for position 1 did not contribute to the subject's score.

The two sketches were then re-covered and the 'photographer' was placed to the right centre side of the layout. The subject remained seated and was asked to think about what a picture would look like if the 'photographer' took a picture from this position. The following sketches were uncovered:

- Sketch 1 - Corresponding to subject's view
- Sketch 4 - Subject's view if leaning in direction of the 'photographer'
- Sketch 7 - Reversal of correct choice
- Sketch 6 - Correct choice.

The subject was asked to choose the sketch which he considered to best represent the picture which the 'photographer' would get. His response was then recorded.

The sketches were re-covered and the 'photographer' was then moved to the left near corner of the layout permitting a diagonal view of the 'mountain scene'. The procedure was continued as when the photographer was in the previous position but for this new position the following sketches were uncovered:

- Sketch 1 - Corresponding to subject's view
- Sketch 8 - Subject's view if leaning in direction of the 'photographer'
- Sketch 4 - Reversal of correct choice
- Sketch 2 - Correct choice.



After the subject had made his choice for this position the sketches were again covered and the 'photographer' was moved to the left centre of the layout. The subject was asked to imagine what kind of a picture the 'photographer' would get and the following sketches were uncovered:

Sketch 1 - Corresponding to subject's view

Sketch 2 - Subject's view if leaning in direction of the 'photographer'

Sketch 6 - Reversal of correct choice

Sketch 3 - Correct choice.

For the final position the 'photographer' was placed opposite the subject; that is, between the back edge of the layout and the bottom of the vertical cardboard containing the sketches. The subject selected from the following sketches:

Sketch 1 - Corresponding to subject's view

Sketch 3 - Subject's view if leaning in direction of the 'photographer'

Sketch 6 - Reversal for this position was sketch 1.  
Sketch 6 was added as the most appropriate fourth choice

Sketch 5 - Correct choice.

The subject remained seated throughout the entire test. The four choices from which the subject has to choose were decided on the following basis - one picture always corresponded to the subject's own viewpoint, another corresponded to the view of the 'mountains' the subject would get if he leaned in the direction of the 'photographer's' position, another represented a left-right and front-back reversal of the correct choice and the fourth represented the correct choice.





Scoring. In arriving at item difficulty each response was coded as 1 if correct, and otherwise 0. However, all responses were recorded and this information will be discussed in dealing with characteristics of individual protocols within the complete sample.

#### Topological and Euclidean Tests - Group 2

This test consisted of items 1, 2, and 3 of the test of Topological Space (Appendix A) and items 1, 2, 3, and 4 of the test of Euclidean Space (Appendix B). Each subject was individually tested in a manner similar to that used in the Topological and Euclidean Tests. The subject was presented with seven geometric shapes on separate 5" by 8" cards. The criterion for success on each item was an accurate representation of each shape by using pieces of plasticine. The subject was provided with an ample supply of strips of plasticine of various lengths and shapes. The setting is illustrated in Figure 5.

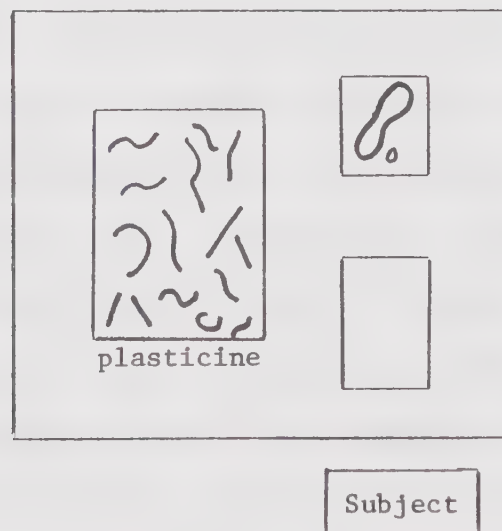


FIGURE 5

SETTING FOR TEST FOR GROUP 2



As a brief training period it was demonstrated to the subject that simple geometric shapes could be made by using plasticine. Included in this demonstration was the shortening of a piece of plasticine which was too long and joining two pieces to form a longer piece. The subject was told he could use as many pieces as he thought necessary. As each shape was completed on a separate sheet of paper it was taken by the investigator, traced with a pencil, and labelled. A different card containing another geometric shape was exposed to the subject. Total testing time was approximately fifteen minutes.

Scoring. Items 1, 2, and 3 of the test of Topological Space were scored in the same way as they were in Group 1 - a score of 1 for each correct response and otherwise 0. For this part of the test the total possible score was 9. Each of the four shapes in the Euclidean Test was scored on five criterion (Appendix B). The total possible score on the test using plasticine was 29.

#### IV. CONTROLS BUILT INTO THE TESTS

In an attempt to minimize the error, the training sessions on all tests were regarded as being very important. In all cases it was clear, before the actual testing began, that the subjects understood the criterion to which they were to respond. The items on each test were given in a generally random order. The nature of the test on Projective Space made it more appropriate to administer consecutively the three items on a rectangular base as well as the three items on the circular base. However, the order in which the different bases were used was alternated. Subjects were encouraged to enter into discussion whenever they indicated any desire to do so. It was



felt that in addition to easing boredom it also provided the subject an opportunity for increased understanding of the initial instructions.

Specific controls were instituted in an effort to ensure the maximum validity and reliability for each particular test used in the study. In the Topological Test of shape representation it was considered important for subjects not to be permitted to move the stimulus cards in any way. Failure to impose this restriction could have resulted in a stimulus of an oblique line being transformed into a horizontal or vertical line. This would have invalidated the initial criterion. Subjects were given twice as many beads as were necessary for accurate representation of the items on 'order'. If the exact number of beads were provided it would have assisted the subject in his response since it would have had strong implications as to when he should discontinue placing the beads on the straight wire.

In the item concerning horizontal and vertical concepts an empty beaker was used to help reinforce the instructions regarding the transformations on the beakers containing water and the plumb line. Also the subjects were asked to draw the water and plumb line as they saw it with the containers in an upright position as well as in a tilted position. This procedure permitted three comparisons of the subject's ability - to draw a vertical and horizontal line when viewed in relation to the Euclidean Plane, to anticipate and represent location of a vertical and horizontal line after transformation in the Euclidean Plane, and to verify this representation.

The stimulus in the Test of Projective Straight Line was to include only the two blocks to represent houses, the matchsticks in clay bases and the rectangular and circular bases. Consequently, the





entire test was administered by placing the rectangular and circular bases on an unpatterned floor. It was felt that if a table top had been used, the perceptual cues presented would have prevented any accurate and meaningful interpretation of the subject's response.

In the test of Co-ordination of Perspectives the four sketches from which the subject was to choose were presented only after the 'picture of the mountains was taken'. This procedure was instituted as a result of a pilot study which indicated that if the sketches were always visible the subject became confused as to exactly what was being photographed.

#### V. ANALYSIS

An item analysis was carried out on the total test of spatial abilities. The internal consistency of the test was assessed by using the Kuder-Richardson Formula 20. Two judges scored the pupils responses. The inter-rater reliability was assessed by subjecting these results to an analysis of variance (Winer, 1962, p. 129).

Relative item difficulty was found using a two-tailed t-test. Relationships between variables are reported using Pearson correlation coefficients. Comparisons of distributions were made by using the Kolmogorov-Smirnov Test for independent samples. The mean scores on the responses of Group 1 and Group 2 were investigated by using a two-tailed t-test and the distributions of those responses were compared by the Kolmogorov-Smirnov Test for independent samples.



## CHAPTER IV

### EVALUATION OF THE INSTRUMENTS AND RESULTS OF THE INVESTIGATION

This chapter reports an evaluation of the instruments used in the present study. It includes inter-rater reliability, and also the internal consistency of the instruments as measured by the Kuder Richardson Formula 20. Correlation coefficients are shown to indicate the relationship between major variables. The results of testing the hypotheses and other tests made on the data are also reported. The analyses reported in this chapter were carried out on the IBM 360/67 computer using Fortran and APL (A Programming Language) on carefully tested programs.

#### I. EVALUATION OF THE INSTRUMENTS

The previous chapter contained a description of the instruments that were used in this study and a discussion of the way in which they were administered. In this section is reported a post-administration analysis of the instruments using the data from the study.

In order to compare the representation of two-dimensional shapes by using pencil-paper and strips of plasticine it will be recalled that the total sample was divided into two groups. One hundred subjects represented items 1, 2, and 3 from the test of Topological Space (Appendix A) and items 1, 2, 3, and 4 from the test of Euclidean Space (Appendix B) by using pencil and paper. In addition, this same group was administered tests which involved aspects of Topological, Euclidean and Projective Space and a test on Co-ordi-



nation of Perspectives.

The remaining 95 subjects, which constituted Group 2, represented the same shapes as listed above but used strips of plasticine instead of pencil and paper. The distribution of scores on this test is shown in Figure 6. The total possible score on this test was 29. The mean was 21.91, and the standard deviation was 2.43. The reliability of the test as measured by the Kuder Richardson Formula 20 was 0.724. A reliability on the scoring methods of the responses was obtained by considering the independent scores of two judges on all items of this test. The two scores were subjected to an analysis of variance procedure (Winer, 1962). The inter-rater reliability was found to be 0.985.

Intercorrelations between age and the variables on the test which was administered to Group 2 are reported in Table III.

TABLE III  
INTERCORRELATIONS BETWEEN AGE AND SUBTESTS FOR  
TEST OF SPATIAL ABILITIES  
(Group 2, n = 95)

| Variable             | 1     | 2       | 3       | 4       |
|----------------------|-------|---------|---------|---------|
| 1. Age               | 1.000 | 0.588** | 0.698** | 0.702** |
| 2. Topological Items |       | 1.000   | 0.723** | 0.886** |
| 3. Euclidean Items   |       |         | 1.000   | 0.968** |
| 4. Total Test        |       |         |         | 1.000   |

\*\*  
p < .01



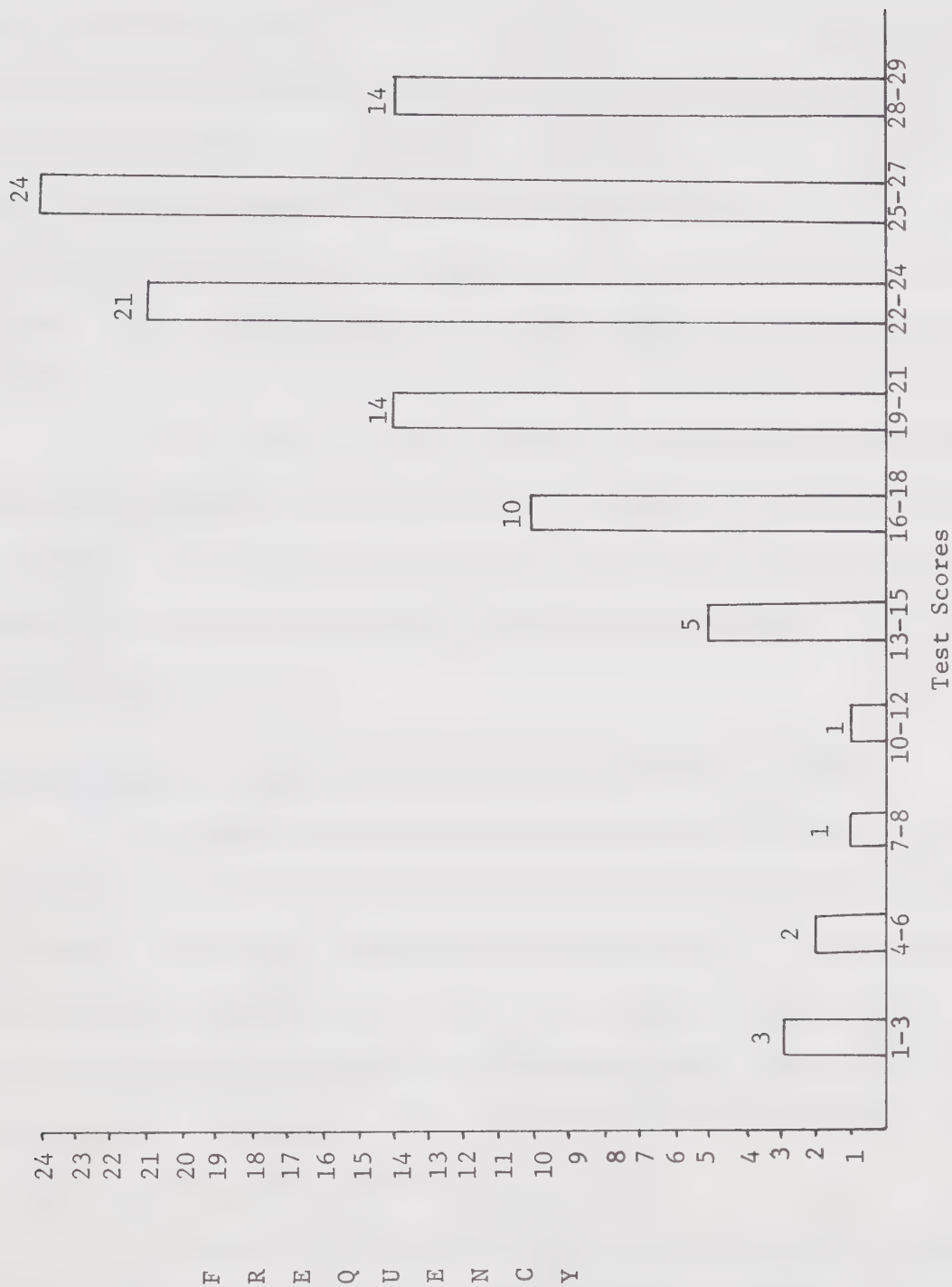


FIGURE 6

DISTRIBUTION OF SCORES ON SPATIAL TEST FOR GROUP 2

(n = 95)





The only purpose in involving Group 2 was to permit a comparison on the responses on two modes of representation - pencil and paper, and plasticine. An analysis of scores for subjects in Groups 1 and 2 are shown in Table IV. A two-tailed t-test on the means of each test and on each property within each test indicated no significant differences between the two groups. A Kolmogorov-Smirnov Test showed no significant difference in the distribution between the two groups when the total scores were considered. (The obtained deviation was 0.0789. The required deviation for significance at the .05 level is 0.191.)

On the basis of these analyses it is concluded that pencil-paper and plasticine are equally effective media in permitting subjects to exhibit their spatial abilities as investigated in the present study. Hence, no further analysis will be done on the responses of the subjects in Group 2.

#### Evaluation of the Total Tests of Spatial Abilities - Group 1

The range, mean, standard deviation and kurtosis for the total test of spatial abilities is reported in Table V. The distributions of scores on the total tests of spatial abilities is shown in Figure 7. A total test reliability of 0.796, as calculated using the Kuder Richardson Formula 20, was considered to be high enough to accept the scores on the test as indicative of the level of spatial development of the subjects in the present study.

An inter-rater reliability on the scoring of the responses was obtained by using an analysis of variance procedure on independent scoring by two judges on the geometric shapes which subjects were



TABLE IV

RANGE, MEAN AND STANDARD DEVIATION FOR TOPOLOGICAL AND  
EUCLIDEAN TESTS AND THEIR CATEGORIES

| Test                   | Group 1 |       |      | Group 2 |       |      | t-values | Probability |
|------------------------|---------|-------|------|---------|-------|------|----------|-------------|
|                        | Range   | Mean  | S.D. | Range   | Mean  | S.D. |          |             |
| Total                  |         |       |      |         |       |      |          |             |
| Topological            | 4-9     | 7.98  | 1.45 | 0-9     | 7.77  | 2.43 | 0.737    | > .05       |
| Topological Closure    | 0-3     | 2.75  | 0.57 | 0-3     | 2.72  | 0.79 | 0.305    | > .05       |
| Topological Separation | 1-3     | 2.59  | 0.66 | 0-3     | 2.48  | 0.89 | 0.984    | > .05       |
| Topological Proximity  | 0-3     | 2.64  | 0.69 | 0-3     | 2.57  | 0.89 | 0.616    | > .05       |
| Total Euclidean        | 2-20    | 14.86 | 3.70 | 1-20    | 14.14 | 4.21 | 1.270    | > .05       |
| Closure                | 2-4     | 3.83  | 0.45 | 0-4     | 3.64  | 0.89 | 1.895    | > .05       |
| Presence of Angles     | 0-4     | 3.65  | 0.80 | 0-4     | 3.44  | 0.93 | 1.693    | > .05       |
| Presence of Lines      | 0-4     | 3.67  | 0.79 | 0-4     | 3.49  | 0.83 | 1.552    | > .05       |
| Accuracy of Angles     | 0-4     | 2.15  | 1.19 | 0-4     | 1.93  | 1.32 | 1.224    | > .05       |
| Accuracy of Lines      | 0-4     | 1.56  | 1.29 | 0-4     | 1.64  | 1.40 | -0.415   | > .05       |



TABLE V

RANGE, MEAN, STANDARD DEVIATIONS, SKEWNESS AND KURTOSIS ON  
TEST OF SPATIAL ABILITIES FOR GROUP 1

| Test                             | Possible<br>Range | Mean  | Standard<br>Deviation | Skewness | Kurtosis |
|----------------------------------|-------------------|-------|-----------------------|----------|----------|
| Total                            | 0-49              | 34.33 | 8.11                  | -0.164   | 0.074    |
| Topological                      | 0-17              | 14.02 | 2.80                  | -1.017   | 0.596    |
| Euclidean                        | 0-22              | 15.37 | 4.09                  | -0.859   | 0.408    |
| Projective                       | 0-6               | 3.82  | 1.95                  | -0.399   | -1.123   |
| Co-ordination<br>of Perspectives | 0-4               | 1.12  | 0.92                  | 0.303    | -0.865   |





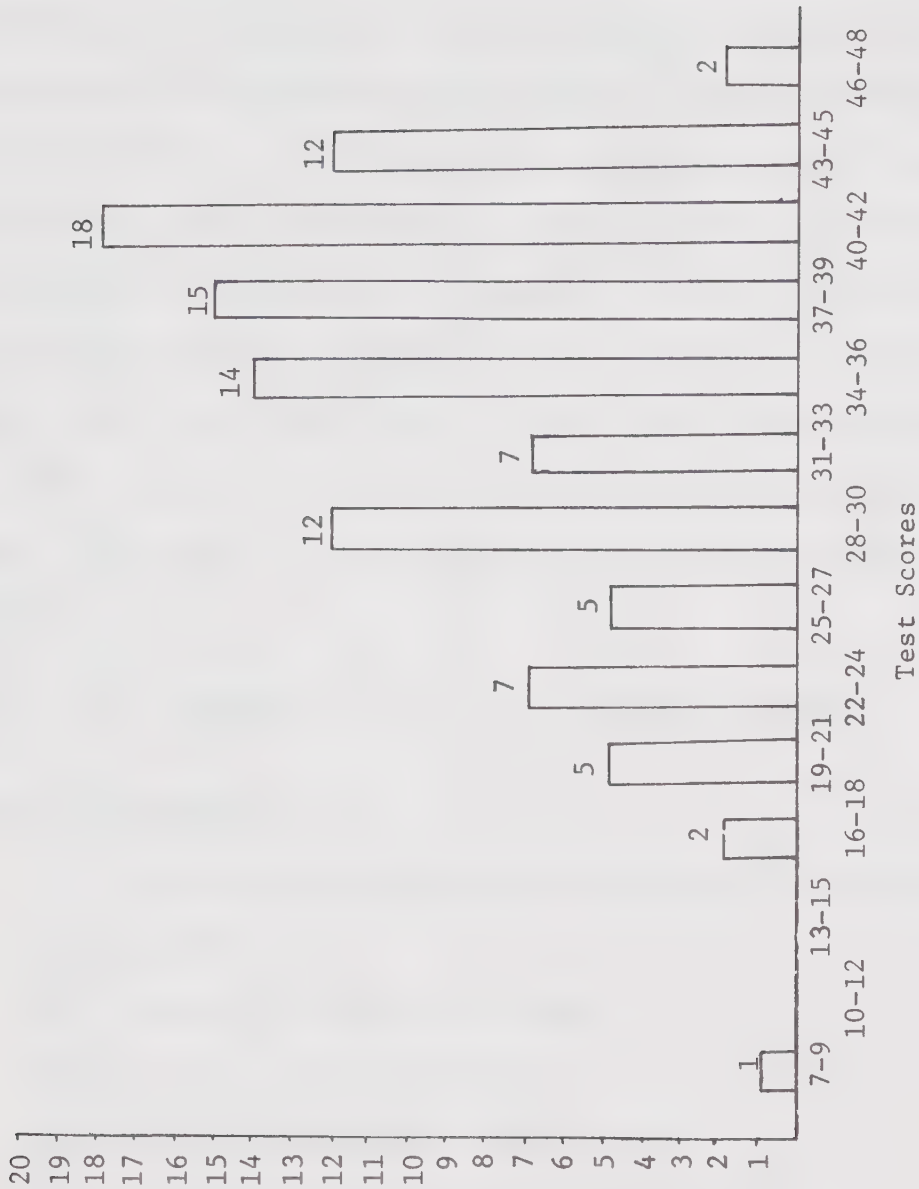


FIGURE 7  
DISTRIBUTION OF SCORES ON TOTAL TEST GROUP 1  
(n = 100)



required to draw. Using this procedure (Winer, 1962) the inter-rater reliability for Group 1 was 0.989.

Intercorrelations between the subtests of the total test are reported in Table VI. The significant correlations between the various subtests are not surprising in view of the fact that all subtests were assumed to be measuring some aspects of spatial abilities. Hence, these correlations might be regarded as some indication of the validity of the total test. The variable of age correlates significantly with all subtests. This finding is important in that it may indicate that the field of spatial abilities is a developmental phenomenon. However, it is a very gross finding since it gives little information on the kinds of spatial abilities possessed by subjects in various age groups. This aspect will be investigated in another section of this chapter.

## II. ANALYSIS OF TOTAL TEST OF SPATIAL ABILITIES

### Results of Hypothesis One

There is no significant difference between the mean scores for boys and girls on:

- (a) the test of Topological Space;
- (b) the test of Euclidean Space;
- (c) the test of Projective Space;
- (d) the test of Co-ordination of Perspectives.

Results. The mean scores for boys and girls on each test are reported in Table VII. All parts of hypothesis one was tested by using a two-tailed t-test. It was found that the means on all the tests were not significantly different for boys and girls. Further



TABLE VI

INTERCORRELATIONS BETWEEN AGE AND SUBTESTS OF TOTAL TEST OF  
SPATIAL ABILITIES

(Group 1, n = 100)

| Variable                         | 1     | 2       | 3       | 4       | 5      | 6       |
|----------------------------------|-------|---------|---------|---------|--------|---------|
| 1. Age                           | 1.000 | 0.638** | 0.620** | 0.678** | 0.243* | 0.714** |
| 2. Topological Test              |       | 1.000   | 0.730** | 0.682** | 0.112  | 0.889** |
| 3. Euclidean Test                |       |         | 1.000   | 0.587** | 0.236* | 0.924** |
| 4. Projective Test               |       |         |         | 1.000   | 0.216* | 0.796** |
| 5. Co-ordination of Perspectives |       |         |         |         | 1.000  | 0.322** |
| 6. Total Test                    |       |         |         |         |        | 1.000   |

\* p < .05

\*\* p < .01



TABLE VII  
COMPARISON OF MEAN SCORES FOR BOYS AND GIRLS ON TESTS OF  
SPATIAL ABILITIES  
(n = 100)

| No.               |       | Topological<br>Space | Euclidean<br>Space | Projective<br>Space | Co-ordination<br>Perspectives |       |
|-------------------|-------|----------------------|--------------------|---------------------|-------------------------------|-------|
| S                 | Girls | 45                   | 14.200             | 15.511              | 3.778                         | 1.133 |
| E                 |       |                      |                    |                     |                               |       |
| X                 | Boys  | 55                   | 13.818             | 15.564              | 3.855                         | 1.109 |
|                   |       |                      |                    |                     |                               |       |
| t-values          |       | 0.655                | -0.0673            | -0.1964             | 0.1299                        |       |
| Probability Level |       | > .05                | > .05              | > .05               | > .05                         |       |





analysis on the means on each category within each test also indicated no significant differences for boys and girls.

Conclusion. All parts of Hypothesis One were accepted.

#### Analysis of Subtests of Total Test of Spatial Abilities

This section contains the analysis of the results of each subtest in the total test of spatial abilities - Topological, Euclidean, Projective Space and the test of Co-ordination of Perspectives. The first part of the report on each subtest is devoted to a comparative analysis of the categories in general. The second part is derived from this analysis and distributes the protocols into a series of divisions related to the objectification of the subject's responses in each subtest.

#### TEST OF TOPOLOGICAL SPACE

##### General Analysis

The test of Topological Space was administered to Group 1 and had a total possible score of 17. The mean on the test was 14.02 and the standard deviation was 2.80. Figure 8 illustrates the criteria of failure and success on each of the categories of closure, separation, and proximity. A shape determined by a continuous line or lines was scored a success on the category of closure. Proximity was determined by using a four part grid. The criterion for success on the category of separation is described in detail in Appendix A. Intercorrelations between age and variables on the Topological Test are reported in Table VIII which indicates that age is significantly related with all variables. This finding will be further investigated in analyzing the



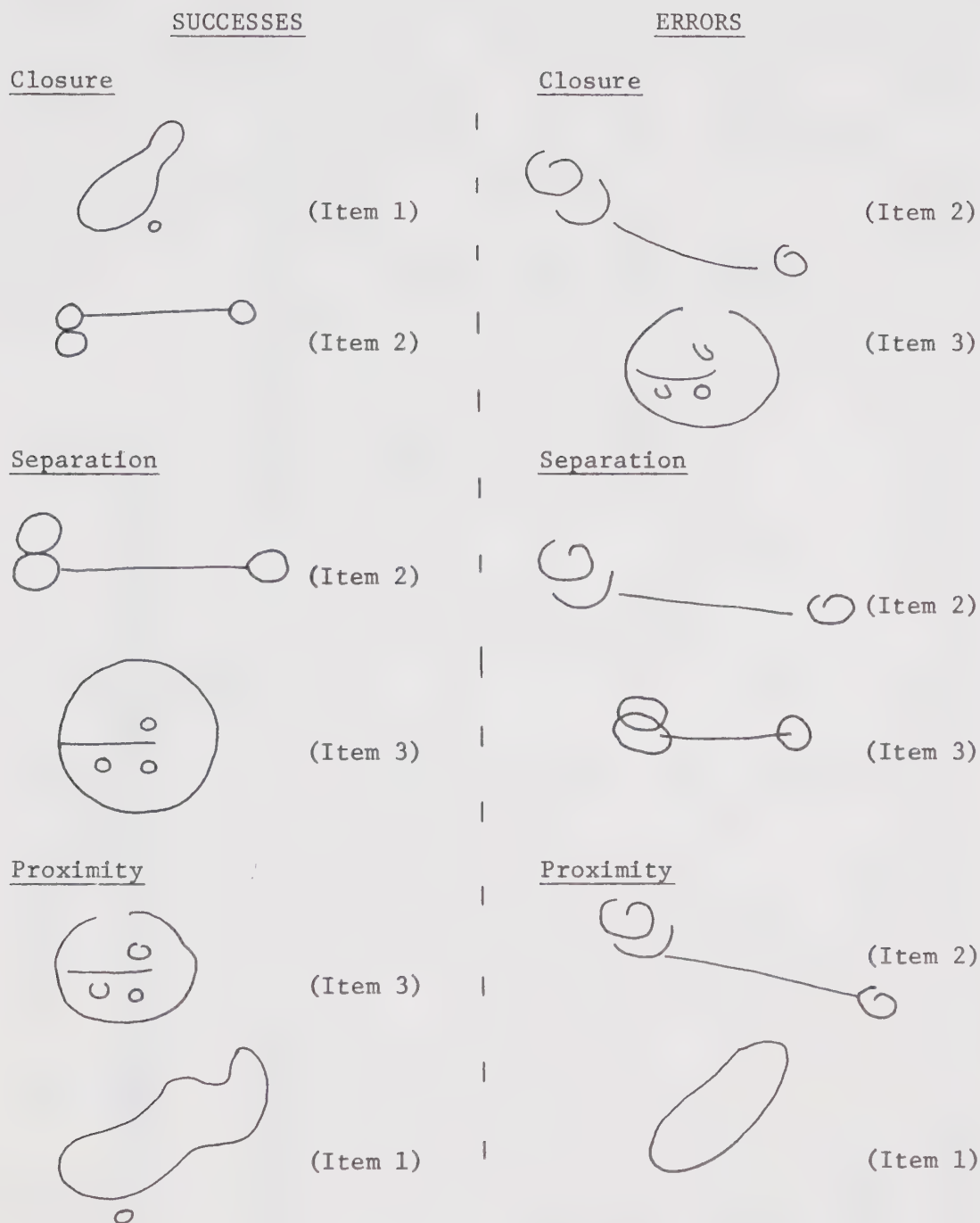


FIGURE 8

EXAMPLES OF RESPONSES CONSIDERED AS SUCCESSES AND AS ERRORS IN  
THREE ITEMS IN THE TEST OF TOPOLOGICAL SPACE



TABLE VIII  
INTERCORRELATIONS BETWEEN AGE AND VARIABLES ON THE  
TOPOLOGICAL TEST

(Group 1, n = 100)

| Variable                  | 1     | 2       | 3       | 4       | 5       | 6       | 7       |
|---------------------------|-------|---------|---------|---------|---------|---------|---------|
| 1. Age                    | 1.000 | 0.313** | 0.406** | 0.502** | 0.446** | 0.545** | 0.638** |
| 2. Topological Closure    |       | 1.000   | 0.414** | 0.306** | 0.066   | 0.339** | 0.538** |
| 3. Topological Separation |       |         | 1.000   | 0.641** | 0.275** | 0.407** | 0.710** |
| 4. Topological Proximity  |       |         |         | 1.000   | 0.412** | 0.476** | 0.773** |
| 5. Order (Item 4)         |       |         |         |         | 1.000   | 0.544** | 0.650** |
| 6. Order (Item 5)         |       |         |         |         |         | 1.000   | 0.841** |
| 7. Total Topological Test |       |         |         |         |         |         | 1.000   |

\*\*  
p < .01



responses by age category. Table IX shows the distribution of topological success and the number of topological errors on each category for each age group.

### Results of Hypothesis Two

There is no significant difference between

- (a) the mean scores on each of the categories of Closure, Separation, and Proximity on the test of Topological Space.
- (b) the distributions on each division on the developmental scale for the test of Topological Space.

Results. The first analysis centered directly on the categories themselves in an effort to determine their nature and relative difficulty. Examination of the results reveals that the first three categories of closure, separation and proximity as reported in Table IX are of appreciably equivalent difficulty. A two-tailed t-test indicated that none of the first three categories are significantly more difficult than the other. The mean scores on the categories of ordering from a circle and ordering from a 'figure-eight' configuration were significantly different beyond the .01 level. The t-values and probability levels are reported in Table X.

The various divisions on the development scale were devised on the basis of category difficulty. The scale has four divisions for the six age categories over which the total subjects in Group 1 are distributed. Subjects included in the various divisions are as follows:

- Division 1 - Subjects who were not successful on any of closure, separation, proximity and order.
- Division 2 - Subjects who were successful on closure, separation and proximity but were not successful on ordering from a 'figure-eight' configuration.





TABLE IX  
DISTRIBUTION OF TOPOLOGICAL PROPERTIES BY AGE CATEGORY

| Age<br>(Months)             | No. | Total<br>Errors | Closure<br>Success | Separation<br>Success | Proximity<br>Success | Order<br>(Circle) | Order<br>(Figure<br>Eight) |
|-----------------------------|-----|-----------------|--------------------|-----------------------|----------------------|-------------------|----------------------------|
| 45-54                       | 4   | 29              | 9                  | 9                     | 4                    | 13                | 4                          |
| 55-64                       | 15  | 91              | 37                 | 31                    | 33                   | 41                | 4                          |
| 65-74                       | 34  | 128             | 90                 | 86                    | 87                   | 118               | 69                         |
| 75-84                       | 23  | 30              | 68                 | 64                    | 68                   | 90                | 71                         |
| 85-94                       | 20  | 19              | 59                 | 59                    | 59                   | 80                | 64                         |
| 95-103                      | 4   | 7               | 12                 | 10                    | 11                   | 16                | 12                         |
| Total                       | 100 | 304             | 275                | 259                   | 262                  | 358               | 242                        |
| Percentage<br>of<br>Success |     |                 | 91.6               | 86.5                  | 87.3                 | 89.5              | 60.5                       |



TABLE X  
t-VALUES AND PROBABILITY LEVELS FOR CATEGORIES ON  
TOPOLOGICAL TEST

| Category                        | 1 | 2     | 3      | 4 | 5        |
|---------------------------------|---|-------|--------|---|----------|
| 1. Closure                      | - | 2.362 | 1.465  | - | -        |
| 2. Separation                   |   | -     | -0.869 | - | -        |
| 3. Proximity                    |   |       | -      | - | -        |
| 4. Order from<br>Circle         |   |       |        | - | 11.219** |
| 5. Order from<br>'Figure Eight' |   |       |        |   | -        |

\*\*Probability Level < .01



Division 3A - Subjects who obtained two or three out of a possible maximum of four on ordering from a 'figure-eight' shaped configuration.

Division 3B - Subjects who were completely accurate on ordering from a 'figure-eight' shaped configuration.

This distribution is shown in Table XI.

Because the category of ordering from a circle was not significantly more difficult than the categories of closure, separation and proximity it was not included in devising the developmental scale. However, all subjects who had complete success on ordering from a 'figure-eight' configuration also had complete success on ordering from a circle. Ninety-five percent of the subjects in Division 3A fulfilled conditions necessary for Divisions 1 and 2. The distributions in Table XI were tested using the Kolmogorov-Smirnov test for independent samples. The results of this investigation is reported in Table XII.

Conclusions. On the basis of these findings Hypothesis 2(a) was accepted. However, it may be that definite conclusions regarding the relative difficulty of closure, separation and proximity cannot be made from this test. The shapes in the test items were probably not appropriate for such scoring categories as separation and proximity. A rather superficial study of the responses to the 'draw a man' exercise used in the training period to the test suggested that this drawing was more suitable to a study of separation and proximity. However, the responses on this item were not analyzed. It was included only as a training item.



TABLE XI  
DISTRIBUTION OF SUBJECTS BY AGE AND DIVISION OF TEST  
OF TOPOLOGICAL SPACE

| Age<br>(Months) | No. | Division 1 | Division 2 | Division 3A | Division 3B |
|-----------------|-----|------------|------------|-------------|-------------|
| 45-54           | 4   | 4          | 0          | 0           | 0           |
| 55-64           | 15  | 5          | 6          | 4           | 0           |
| 65-74           | 34  | 5          | 8          | 12          | 9           |
| 75-84           | 23  | 0          | 1          | 10          | 12          |
| 85-94           | 20  | 0          | 0          | 10          | 10          |
| 95-103          | 4   | 0          | 0          | 1           | 3           |
| Total           | 100 | 14         | 15         | 37          | 34          |





TABLE XII  
KOLMOGOROV-SMIRNOV  
DEVIATIONS FOR DISTRIBUTION ON TEST OF TOPOLOGICAL SPACE

|             | Division 2 |                         | Division 3A |                         | Division 3B |                         |
|-------------|------------|-------------------------|-------------|-------------------------|-------------|-------------------------|
|             | Obtained   | (Required<br>.05 Level) | Obtained    | (Required<br>.05 Level) | Obtained    | (Required<br>.05 Level) |
| Division 1  | 0.5238     | 0.6768                  | 0.5676**    | 0.4267                  | 0.7353**    | 0.4319                  |
| Division 2  |            |                         | 0.4426*     | 0.4069                  | 0.5321**    | 0.3769                  |
| Division 3A |            |                         |             |                         | 0.19        | 0.4142                  |
| Division 3B |            |                         |             |                         | -           | -                       |

\*  
p < .05

\*\*  
p < .01



Hypothesis 2(b) was rejected. There were significant differences in distributions of subjects who were unsuccessful on all categories of closure, proximity, separation and order and subjects who were successful on ordering from a 'figure-eight' configuration. There were significant differences also in the distribution of subjects who were successful on the topological categories of closure, proximity and separation but were unsuccessful on ordering from a 'figure-eight' and those subjects who were successful on ordering from a 'figure-eight'. The differences in distributions were in the direction of indicating a greater degree of success for older subjects. A discussion of these differences is included in Chapter V.

## TEST OF EUCLIDEAN SPACE

### General Analysis

The test of Euclidean Space (Appendix B) involved the representation of geometric shapes as well as an investigation into the child's ability to anticipate states arising from transformation of a vertical and horizontal nature in the Euclidean plane. The total possible score was 22. The mean on this test was 15.37 and the standard deviation was 4.09. Details of the scoring are outlined in Appendix B. The criteria of errors and success on each scoring category are illustrated in Figure 9.

Table XIII reports correlations between age and variables on the Euclidean Test. Almost all correlations are significant. It is apparent that if hypotheses were tested which required relationships of the kind indicated by correlations, the conclusions would have to be regarded as very gross and indefinite. The correlations in Table XIII



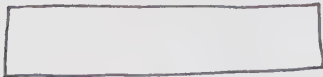
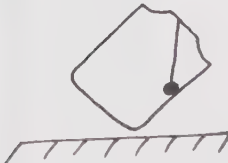
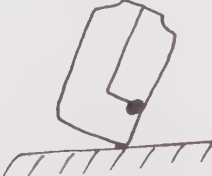
| <u>SUCCESSSES</u>                 |                                                                                     |          | <u>ERRORS</u>                                                                                                             |          |
|-----------------------------------|-------------------------------------------------------------------------------------|----------|---------------------------------------------------------------------------------------------------------------------------|----------|
| <u>Closure</u>                    |    | (Item 2) | <u>Closure</u><br>                       | (Item 3) |
| <u>Presence of Angles</u>         |    | (Item 4) | <u>Presence of Angles</u><br>           | (Item 3) |
| <u>Presence of Straight Lines</u> |    | (Item 4) | <u>Presence of Straight Lines</u><br>   | (Item 3) |
| <u>Accuracy of Angles</u>         |    | (Item 2) | <u>Accuracy of Angles</u><br>           | (Item 4) |
| <u>Accuracy of Straight Lines</u> |  | (Item 3) | <u>Accuracy of Straight Lines</u><br> | (Item 4) |
| <u>Horizontal</u>                 |  | (Item 5) | <u>Horizontal</u><br>                 | (Item 5) |
| <u>Vertical</u>                   |  | (Item 6) | <u>Vertical</u><br>                   | (Item 6) |

FIGURE 9

EXAMPLES OF RESPONSES CONSIDERED AS SUCCESSES AND AS ERRORS  
ON THE TEST OF EUCLIDEAN SPACE



TABLE XIII

INTERCORRELATIONS BETWEEN AGE, CLOSURE AND VARIABLES ON EUCLIDEAN TEST

| Variable                         | 1     | 2       | 3       | 4       | 5       | 6       | 7       | 8       | 9       |
|----------------------------------|-------|---------|---------|---------|---------|---------|---------|---------|---------|
| 1. Age                           | 1.000 | 0.298** | 0.422** | 0.418** | 0.515** | 0.365** | 0.521** | 0.596** | 0.620** |
| 2. Closure                       |       | 1.000   | 0.306** | 0.322** | 0.290** | 0.251*  | 0.068   | 0.215*  | 0.441** |
| 3. Presence<br>of<br>Angles      |       |         | 1.000   | 0.985** | 0.512** | 0.429** | 0.241*  | 0.398** | 0.659** |
| 4. Presence<br>of Lines          |       |         |         | 1.000   | 0.499** | 0.427** | 0.226*  | 0.392** | 0.662** |
| 5. Accuracy<br>of<br>Angles      |       |         |         |         | 1.000   | 0.672** | 0.328** | 0.389** | 0.702** |
| 6. Accuracy<br>of Lines          |       |         |         |         |         | 1.000   | 0.237*  | 0.359** | 0.659** |
| 7. Horizontal<br>(Item 5)        |       |         |         |         |         |         | 1.000   | 0.382** | 0.607** |
| 8. Vertical<br>(Item 6)          |       |         |         |         |         |         |         | 1.000   | 0.638** |
| 9. Total on<br>Euclidean<br>Test |       |         |         |         |         |         |         |         | 1.000   |

\* p &lt; .05

\*\* p &lt; .01





indicate the following:

- The development of Euclidean aspects of spatial abilities is directly related to age.
- Generally, the various aspects of the Euclidean Test are significantly correlated.

Table XIV was derived from successes and errors on each category. The analysis associated with the data contained in this table centers on the categories themselves and makes no attempt to analyze any characteristics of the sample or any individual protocols. On the basis of this analysis a scale was devised to look at more specific characteristics of the various test categories.

### Results of Hypothesis Three

There is no significant difference between

- (a) the mean scores on each of the categories of Closure, Presence of Angles, Presence of Straight Lines, Accuracy of Angles, Accuracy of Straight Lines in the test of Euclidean Space.
- (b) the distributions on each division on the developmental scale for the test of Euclidean Space.

Results. The relative difficulty of the categories in the test of Euclidean Space which is reported in Table XIV was investigated by a two-tailed t-test. These results are reported in Table XV.

The results indicate that the categories of closure, presence of angles, and presence of straight lines were not significantly different from each other in difficulty. However, the difficulty of the categories requiring accuracy of straight lines and accuracy of angles were significantly more difficult than the first three categories.



TABLE XIV  
DISTRIBUTION OF EUCLIDEAN PROPERTIES BY AGE CATEGORY

| Age<br>(Months)          | No. | Total<br>Errors | Closure<br>Success | Presence<br>of<br>Angles | Presence<br>of<br>Lines | Accuracy<br>of<br>Angles | Accuracy<br>of<br>Lines | Horizontal<br>Success | Vertical<br>Success |
|--------------------------|-----|-----------------|--------------------|--------------------------|-------------------------|--------------------------|-------------------------|-----------------------|---------------------|
| 45-54                    | 4   | 46              | 14                 | 11                       | 11                      | 5                        | 1                       | 0                     | 0                   |
| 55-64                    | 15  | 166             | 50                 | 44                       | 44                      | 13                       | 11                      | 1                     | 1                   |
| 65-74                    | 34  | 243             | 130                | 119                      | 121                     | 71                       | 52                      | 7                     | 5                   |
| 75-84                    | 23  | 104             | 89                 | 92                       | 92                      | 58                       | 47                      | 10                    | 14                  |
| 85-94                    | 20  | 78              | 80                 | 80                       | 80                      | 59                       | 38                      | 11                    | 14                  |
| 95-103                   | 4   | 24              | 16                 | 16                       | 10                      | 10                       | 7                       | 2                     | 3                   |
| Total                    | 100 | 661             | 379                | 362                      | 358                     | 216                      | 156                     | 31                    | 37                  |
| Percentage<br>of Success |     |                 | 94.7               | 90.5                     | 89.5                    | 54.0                     | 39.0                    | 31.0                  | 37.0                |



TABLE XV

t-VALUES AND PROBABILITY LEVELS FOR CATEGORIES ON EUCLIDEAN TEST

| Category                           | 1 | 2     | 3      | 4        | 5        | 6 | 7      |
|------------------------------------|---|-------|--------|----------|----------|---|--------|
| 1. Closure                         | - | 2.261 | 2.065  | 14.562** | 17.985** | - | -      |
| 2. Presence of Angles              |   | -     | -1.421 | 14.295** | 17.439** | - | -      |
| 3. Presence of Straight Lines      |   |       | -      | 14.357** | 17.731** | - | -      |
| 4. Accuracy of Angles              |   |       |        | -        | 5.807**  | - | -      |
| 5. Accuracy of Straight Lines      |   |       |        |          | -        | - | -      |
| 6. Horizontal (Water in Beaker)    |   |       |        |          |          | - | -1.136 |
| 7. Vertical (Plumb Line in Bottle) |   |       |        |          |          |   | -      |

\*\*  
p < .01



On the basis of these investigations a developmental scale was devised. The following criterion was used to assign subjects to the divisions on the scale:

- Division 1 - Subjects who did not attain success on two of the three categories of closure, presence of angles and presence of lines. (Success on each category was taken as a score of two or more. The total possible score on each of the three categories was four.)
- Division 2 - Subjects who were successful in two of the three categories mentioned above but were unsuccessful in the accuracy of angles, accuracy of straight lines, horizontal and vertical categories.
- Division 3 - Subjects who were successful on the categories of accuracy of lines and accuracy of angles.
- Division 4 - Subjects who were successful on the horizontal and vertical categories.

A subject was always placed in the highest division for which he met the criterion. Ninety-six percent of all subjects who were placed in a certain division obtained the necessary requirements for the divisions which preceded it. The distribution of subjects in these divisions is presented in Table XVI.

The distributions over age categories were tested using the Kolmogorov-Smirnov Test for independent samples. The results of the application of the test are summarized in Table XVII.

Conclusion. Hypothesis 3(a) was rejected. There were no significant differences in the mean scores on the categories of closure, presence of angles and presence of straight lines. However, the means on these three categories were all significantly different from the means on the categories of accuracy of angles and accuracy of straight lines.





TABLE XVI  
DISTRIBUTION OF SUBJECTS BY AGE AND DIVISION ON TEST OF  
EUCLIDEAN SPACE

| Age<br>(Months) | Number | Division 1 | Division 2 | Division 3 | Division 4 |
|-----------------|--------|------------|------------|------------|------------|
| 45-54           | 4      | 1          | 3          | 0          | 0          |
| 55-64           | 15     | 1          | 11         | 2          | 1          |
| 65-74           | 34     | 3          | 16         | 8          | 7          |
| 75-84           | 23     | 0          | 4          | 8          | 11         |
| 85-94           | 20     | 0          | 1          | 3          | 16         |
| 95-103          | 4      | 0          | 0          | 0          | 4          |
| Total           | 100    | 5          | 35         | 21         | 39         |



TABLE XVII  
KOLMOGOROV-SMIRNOV  
DEVIATIONS FOR DISTRIBUTIONS ON TEST OF EUCLIDEAN SPACE

|            | Division 2 |                         | Division 3          |                         | Division 4           |                         |
|------------|------------|-------------------------|---------------------|-------------------------|----------------------|-------------------------|
|            | Obtained   | (Required<br>.05 Level) | Obtained            | (Required<br>.05 Level) | Obtained             | (Required<br>.05 Level) |
| Division 1 | 0.1429     | 0.6502                  | 0.5238 <sup>*</sup> | 0.4768                  | 0.7949 <sup>**</sup> | 0.6460                  |
| Division 2 |            |                         | 0.3810 <sup>*</sup> | 0.3754                  | 0.652 <sup>**</sup>  | 0.3167                  |
| Division 3 |            |                         |                     |                         | 0.37 <sup>*</sup>    | 0.3681                  |
| Division 4 |            |                         |                     |                         | -                    | -                       |

\*  $p < .05$

\*\*  $p < .01$



Hypothesis 3(b) was rejected. There was a significant difference in the distribution of subjects who failed on all categories and subjects who were successful on the categories involving horizontal and vertical representation. The distribution of subjects who were successful on the categories involving horizontal and vertical concepts was also significantly different from that group of subjects who were unsuccessful on those concepts but were successful in the categories of accuracy of lines and accuracy of angles. The distribution indicated that older subjects responded with a greater degree of accuracy. A more detailed discussion of this finding is reported in Chapter V.

## TEST OF PROJECTIVE SPACE

### General Analysis

The test of Projective Space was restricted in this study to the projective straight line. Details of the test and the administrative procedure are contained in Appendix C. The total possible score on this test was 6. For the sample in this study the mean was 3.82, and the standard deviation was 1.95. The great variety of responses made it impractical to use a continuum in scoring. The test items were scored as correct, a score of 1, or incorrect, a score of 0, depending on whether or not the subject fulfilled the criterion of using the materials provided to form a straight line path between two objects whose positions were predetermined. Figure 10 illustrates varieties of responses which were scored as successes and errors.

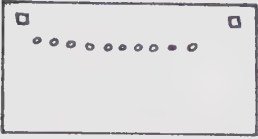
Intercorrelations between age and variables on the test of Projective Space are reported in Table XVIII. Problems having common



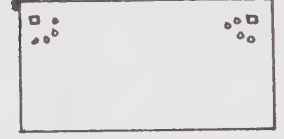
SUCCESSSES

ERRORS

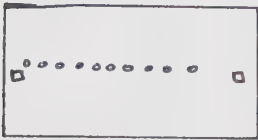
Problem 1



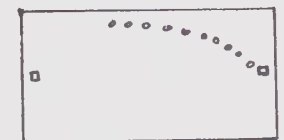
Problem 1



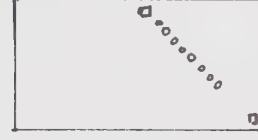
Problem 2



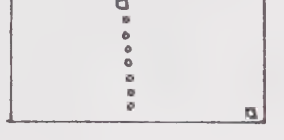
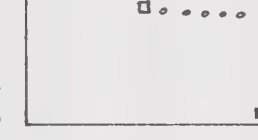
Problem 2



Problem 3



Problem 3



Problem 4



Problem 4



Problem 5



Problem 5



Problem 6



Problem 6



FIGURE 10

EXAMPLES OF RESPONSES CONSIDERED AS SUCCESSES AND AS ERRORS  
ON THE TEST OF PROJECTIVE SPACE





TABLE XVIII  
INTERCORRELATIONS BETWEEN AGE AND VARIABLES ON THE TEST  
OF PROJECTIVE SPACE

| Variable                                            | 1     | 2       | 3       | 4       | 5       |
|-----------------------------------------------------|-------|---------|---------|---------|---------|
| 1. Age                                              | 1.000 | 0.597** | 0.533** | 0.580** | 0.678** |
| 2. Problems 1 and 4<br>(Lines near edge<br>of base) |       | 1.000   | 0.659** | 0.532** | 0.845** |
| 3. Problems 2 and 5<br>(Central Lines)              |       |         | 1.000   | 0.492** | 0.843** |
| 4. Problems 3 and 6<br>(Oblique Lines)              |       |         |         | 1.000   | 0.831** |
| 5. Total on Pro-<br>jective Test                    |       |         |         |         | 1.000   |

\*\* p < .01



characteristics have been grouped to form a total of three categories. Problems 1 and 4 involve constructing a straight line near the edge of a given base and hence, it is presumed, is subjected to strong perceptual cues. Problems 2 and 5 require a straight line to be constructed but somewhat removed from the perceptual cues which were present in the two previous problems. Oblique lines are required in Problems 3 and 6.

The correlations reported here are consistent with those reported on the two previous tests of Topological and Euclidean Space. Age is significantly related to all categories of the test. However, the correlations indicate nothing about the relative difficulty of the various test items nor how subjects of various age groups responded to individual problems.

#### Results of Hypothesis Four

There is no significant difference between

- (a) the mean scores on each of the categories of Line Near the Edge, Central Line and Oblique Line on the test of Projective Space.
- (b) the distributions on each division on the developmental scale for the test of Projective Space.

Results. Table XIX reports a general investigation into the six items on the test of Projective Space, and indicates the successes and errors for each age category. If the problems not concerning obliques (Problems 1, 2, 4 and 5) are compared with the obliques (Problems 3 and 6) it is found by a correlated t-test that the relative difficulty of problems of lines near the edge of the base (1 and 4)



TABLE XIX

DISTRIBUTION FOR AGE CATEGORIES FOR PROBLEMS ON TEST OF PROJECTIVE SPACE

| Age<br>(Months) | No. | Errors | Rectangular Base                   |                                  |                      | Circular Base                      |                                  |                      |
|-----------------|-----|--------|------------------------------------|----------------------------------|----------------------|------------------------------------|----------------------------------|----------------------|
|                 |     |        | 1<br>Straight<br>Line<br>Near Edge | 2<br>Central<br>Straight<br>Line | 3<br>Oblique<br>Line | 1<br>Straight<br>Line<br>Near Edge | 2<br>Central<br>Straight<br>Line | 3<br>Oblique<br>Line |
| 45-54           | 4   | 19     | 2                                  | 1                                | 0                    | 0                                  | 2                                | 0                    |
| 55-64           | 15  | 64     | 10                                 | 8                                | 0                    | 4                                  | 2                                | 2                    |
| 65-74           | 34  | 98     | 26                                 | 22                               | 14                   | 15                                 | 21                               | 8                    |
| 75-84           | 23  | 28     | 22                                 | 20                               | 13                   | 22                                 | 19                               | 14                   |
| 85-94           | 20  | 15     | 20                                 | 19                               | 17                   | 18                                 | 18                               | 13                   |
| 95-103          | 4   | 2      | 4                                  | 4                                | 3                    | 4                                  | 4                                | 3                    |
| Total           | 100 | 226    | 84                                 | 74                               | 47                   | 63                                 | 66                               | 40                   |



are not significantly different from problems on central straight lines (2 and 5) but both groups (Problems 1 and 4 and Problems 2 and 5) are significantly different from combined problems on oblique lines (3 and 6). This information is reported in Table XX.

TABLE XX  
t-VALUES AND PROBABILITY LEVELS FOR CATEGORIES  
ON PROJECTIVE TEST

| Category                                                | 1 | 2     | 3                    |
|---------------------------------------------------------|---|-------|----------------------|
| 1. Lines Near the<br>Edge of Base<br>(Problems 1 and 4) | - | 0.332 | 7.291 <sup>***</sup> |
| 2. Central Lines<br>(Problems 2 and 5)                  |   | -     | 6.506 <sup>***</sup> |
| 3. Oblique Lines<br>(Problems 3 and 6)                  |   |       | -                    |

<sup>\*\*\*</sup>  
p < .001

On the basis of the preceding analysis a developmental scale was devised. Subjects were placed in the divisions on the following criteria:

- Division 1 - Subjects who were unsuccessful on all problems.
- Division 2 - Subjects who were successful on at least one of the problems to be constructed near the edge of base (that is, Problems 1 or 4) but were unsuccessful on all other problems.
- Division 3 - Subjects who were successful on at least one of the problems requiring a central straight line (that is, Problems 2 or 5) but unsuccessful on both problems involving oblique lines (Problems 3 and 6).
- Division 4 - Subjects who were successful on one or both problems requiring an oblique line.

This distribution for each age category is reported in Table XXI.





TABLE XXI

DISTRIBUTION OF SUBJECTS BY AGE AND DIVISION ON TEST OF PROJECTIVE SPACE

| Age<br>(Months) | No. | Division 1                      | Division 2                                              | Division 3                                  | Division 4               |
|-----------------|-----|---------------------------------|---------------------------------------------------------|---------------------------------------------|--------------------------|
|                 |     | Unsuccessful on<br>all problems | Success on one or<br>more problems near<br>edge of base | Success on at<br>least one<br>central prob. | Success on<br>an oblique |
| 45-54           | 4   | 1                               | 1                                                       | 2                                           | 0                        |
| 55-64           | 15  | 2                               | 6                                                       | 5                                           | 2                        |
| 65-74           | 34  | 4                               | 3                                                       | 12                                          | 15                       |
| 75-84           | 23  | 0                               | 0                                                       | 5                                           | 18                       |
| 85-94           | 20  | 0                               | 0                                                       | 3                                           | 17                       |
| 95-103          | 4   | 0                               | 0                                                       | 1                                           | 3                        |
| Total           | 100 | 7                               | 10                                                      | 28                                          | 55                       |



Table XXII compares the scores of subjects on lines near the edge of the bases and central lines with their success on obliques.

TABLE XXII

COMPARISON OF SUBJECT'S SUCCESS ON OBLIQUES AND  
OTHER LINES ON PROJECTIVE TEST

| Oblique | Lines Near the Edge and Central Lines |    |    |    |    | Total |
|---------|---------------------------------------|----|----|----|----|-------|
|         | 0                                     | 1  | 2  | 3  | 4  |       |
| 0       | 7                                     | 10 | 7  | 14 | 7  | 45    |
| 1       |                                       | 0  | 4  | 5  | 11 | 20    |
| 2       |                                       | 0  | 1  | 2  | 32 | 35    |
| Total   | 7                                     | 10 | 12 | 21 | 50 | 100   |

This table indicates that 43 of the 55 subjects who were placed in Division 4 had obtained a perfect score on Problems 1, 2, 4 and 5; that is on the non-oblique lines. A further breakdown indicates that of the 42 subjects whose score was three or less, only 4 of those subjects were successful on any oblique line. Also there was no subject successful on an oblique line who scored less than 3 on the test. Hence, the distributions reported in Table XXI occur with little overlap or inconsistency.



The distributions were tested by the Kolmogorov-Smirnov Test for independent samples. The results are summarized in Table XXIII.

Conclusion. Hypothesis 4(a) was rejected. The findings indicated that the means on items involving lines near the edge of the base and means on central lines were not significantly different from each other but both means were significantly different from the means on the items involving oblique lines.

The distribution of subjects who were successful on central lines was not significantly different from the distribution of subjects who failed on all test items or the distribution of subjects who were successful on lines parallel to the edge of the bases. However, the distribution of subjects who were successful on the obliques was significantly different from all other distributions of subjects in other divisions. Hypothesis 4(b) was, therefore, rejected. The analysis indicated that the older subjects had much more success in constructing oblique lines than the younger subjects.

#### TEST OF CO-ORDINATION OF PERSPECTIVES

##### General Analysis

This test was used to study the development and integration of projective concepts. The test items are explained in detail in Chapter III and in Appendix D. The total possible score on the test was 4. The mean for the sample in this study was 1.12 and the standard deviation was 0.92. Each test item was scored as 1 or 0 depending on whether it was correct or incorrect. However, the characteristics of the incorrect choices were investigated and will be discussed in another section of this chapter.



TABLE XXIII

KOLMOGOROV-SMIRNOV

DEVIATIONS FOR DISTRIBUTIONS ON TEST OF PROJECTIVE SPACE

|                                                                              | Division 2 |                         | Division 3 |                         | Division 4 |                         |
|------------------------------------------------------------------------------|------------|-------------------------|------------|-------------------------|------------|-------------------------|
|                                                                              | Obtained   | (Required<br>.05 Level) | Obtained   | (Required<br>.05 Level) | Obtained   | (Required<br>.05 Level) |
| Division 1<br>(Unsuccessful<br>on all pro-<br>blems)                         | 0.2714     | 0.6232                  | 0.3214     | 0.5747                  | 0.6909**   | 0.5458                  |
| Division 2<br>(Success on at<br>least one pro-<br>blem near edge<br>of base) |            |                         | 0.45       | 0.501                   | 0.6909**   | 0.4675                  |
| Division 3<br>(Success on at<br>least one cen-<br>tral problem)              |            |                         |            |                         | 0.3695*    | 0.3157                  |
| Division 4<br>(Success on<br>an oblique)                                     |            |                         |            |                         | -          | -                       |

\*  $p < .05$ \*\*  $p < .01$





Table XXIV reports intercorrelations between age and the four positions which constitute the test (Appendix D). This correlation matrix illustrates that there are apparent inconsistencies in the results of this test. This is the first subtest of the total test of spatial abilities where age has not been significantly related to the various test items. It should also be noted that in no other subtest was there greater apparent relationships between test items than in this test. Nevertheless, few significant correlations were found.

#### Results of Hypothesis Five

There are no significant differences between

- (a) the mean scores on each of the four positions on the test of Co-ordination of Perspectives;
- (b) the distributions on each division on the developmental scale for the test of Co-ordination of Perspectives.

Results. Table XXV reports the distribution over age categories for each position in the test of Co-ordination of Perspectives. A two-tailed t-test indicated, Table XXVI, that the means on each of the positions to the right of the subject and the left near corner position were significantly different from the means on each of the positions to the left of the subject and opposite the subject. In those last two positions the correct choices (Appendix D) contained only two cones whereas the subject, from the position, was viewing all three cones. The correct choices in the position to the right of the subject and the left near corner position each contained three cones. This indicates that it was more difficult for the subject to imagine a situation with such dissimilarities from his own viewpoint.



TABLE XXIV

INTERCORRELATIONS BETWEEN AGE AND POSITIONS ON TEST OF  
CO-ORDINATION OF PERSPECTIVES

| Variables                          | 1     | 2     | 3     | 4     | 5       | 6       |
|------------------------------------|-------|-------|-------|-------|---------|---------|
| 1. Age                             | 1.000 | 0.011 | 0.073 | 0.158 | 0.276** | 0.243*  |
| 2. Position to right<br>of subject |       | 1.000 | 0.169 | 0.028 | -0.007  | 0.607** |
| 3. Position opposite<br>subject    |       |       | 1.000 | 0.161 | -0.043  | 0.487** |
| 4. Position to left<br>of subject  |       |       |       | 1.000 | 0.075   | 0.520** |
| 5. Left near corner<br>position    |       |       |       |       | 1.000   | 0.542** |
| 6. Total Test                      |       |       |       |       |         | 1.000   |

\*  
p < .05

\*\*  
p < .01



TABLE XXV  
DISTRIBUTIONS FOR AGE CATEGORIES FOR POSITIONS ON TEST OF  
CO-ORDINATION OF PERSPECTIVES

| Age<br>(Months) | No. | Errors | Successes                    |                              |                                 |                             |
|-----------------|-----|--------|------------------------------|------------------------------|---------------------------------|-----------------------------|
|                 |     |        | Position right<br>of subject | Left near corner<br>position | Position<br>opposite<br>subject | Position left<br>of subject |
| 45-54           | 4   | 14     | 0                            | 2                            | 0                               | 0                           |
| 55-64           | 15  | 50     | 6                            | 2                            | 0                               | 2                           |
| 65-74           | 34  | 99     | 15                           | 13                           | 5                               | 4                           |
| 75-84           | 23  | 66     | 12                           | 6                            | 3                               | 5                           |
| 85-94           | 20  | 52     | 8                            | 13                           | 3                               | 4                           |
| 95-103          | 4   | 7      | 3                            | 3                            | 1                               | 2                           |
| Total           | 100 | 288    | 44                           | 39                           | 12                              | 17                          |



TABLE XXVI

t-VALUES AND PROBABILITY LEVELS FOR POSITIONS ON TEST OF  
CO-ORDINATION OF PERSPECTIVES

| Item                            | 1 | 2     | 3       | 4       |
|---------------------------------|---|-------|---------|---------|
| 1. Position right<br>of subject | - | 0.713 | 5.836** | 4.375** |
| 2. Left near corner<br>position |   | -     | 4.496** | 3.692** |
| 3. Position opposite<br>subject |   |       | -       | -1.092  |
| 4. Position left<br>of subject  |   |       |         | -       |

\*\* p < .01





On the basis of item difficulty a developmental scale was devised. Criterion for placement in each of the three divisions of the scale was as follows:

- Division 1 - Subjects who were unsuccessful in all problems.
- Division 2 - Subjects who were successful in their choice on at least one of the positions to the right of the subject and the left near corner. That is, problems whose correct sketches contained the same number of cones as the view of the subject.
- Division 3 - Subjects who were successful on their choice on at least one of the positions to the left of the subject and opposite the subject. That is, problems whose correct sketches contained only two cones whereas the subject, from his viewpoint, could see three cones. Four of the 24 subjects in this division did not reach the requirements for Division 2.

The distribution of the sample in the divisions is reported in Table XXVII.

The distributions in Table XXVII were tested using a Kolmogorov-Smirnov Test for independent proportions. The results are summarized in Table XXVIII.

Conclusion. Hypothesis 5(a) was rejected. The mean scores of the responses on the position of the 'photographer' to the right of the subject and the left near corner position were both significantly different from the mean scores on the position of the 'photographer' to the left of the subject and when placed opposite the subject. That is, the most difficult views for subjects to imagine were those with only two cones visible. The subject could always see three cones from his position.

Hypothesis 5(b) was accepted. The distributions as defined by the developmental scale were not significantly different from each



TABLE XXVII

DISTRIBUTION OF SUBJECTS BY AGE AND DIVISION ON TEST OF  
CO-ORDINATION OF PERSPECTIVES

| Age<br>(Months) | No. | Division 1                       | Division 2                                                                                      | Division 3                                                                                   |
|-----------------|-----|----------------------------------|-------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------|
|                 |     | Unsuccessful on<br>all positions | Successful on at least<br>one of positions right<br>of subject and left<br>near corner position | Successful on at least<br>one of positions to the<br>left and position oppo-<br>site subject |
| 45-54           | 4   | 2                                | 2                                                                                               | 0                                                                                            |
| 55-64           | 15  | 5                                | 8                                                                                               | 2                                                                                            |
| 65-74           | 34  | 13                               | 13                                                                                              | 8                                                                                            |
| 75-84           | 23  | 7                                | 10                                                                                              | 6                                                                                            |
| 85-94           | 20  | 3                                | 11                                                                                              | 6                                                                                            |
| 95-103          | 4   | 0                                | 2                                                                                               | 2                                                                                            |
| Total           | 100 | 30                               | 46                                                                                              | 24                                                                                           |



TABLE XXVIII

KOLMOGOROV-SMIRNOV

DEVIATIONS FOR DISTRIBUTIONS ON TEST OF CO-ORDINATION OF PERSPECTIVES

|            | Division 2 |                         | Division 3 |                         |
|------------|------------|-------------------------|------------|-------------------------|
|            | Obtained   | (Required<br>.05 Level) | Obtained   | (Required<br>.05 Level) |
| Division 1 | 0.1826     | 0.3192                  | 0.25       | 0.3725                  |
| Division 2 |            |                         | 0.1389     | 0.3438                  |
| Division 3 |            |                         | -          | -                       |



other at the .05 level. This indicated that although some positions were significantly more difficult than others, as was reported in Hypothesis 5(a), the responses of the older subjects on those positions were generally no more accurate than the responses of the younger subjects.

### III. ANALYSIS OF INCORRECT RESPONSES, SPECIFIC OBSERVATIONS, OVERVIEW OF THE FINDINGS AND SUMMARY

#### Incorrect Responses

Incorrect responses on the test of Euclidean Space and the test of Co-ordination of Perspectives were investigated regarding their relationship with other responses of the same subject and the overall significance of the responses relative to the development of spatial abilities.

The following discussion centers around the responses of 20 selected subjects. On item 5 of the test of Euclidean Space six of those subjects represented the water as it would appear in a tilted beaker as a horizontal and a vertical line rather than the correct oblique line. (Oblique relative to the sides of the beaker.) This is illustrated in Figure 11.

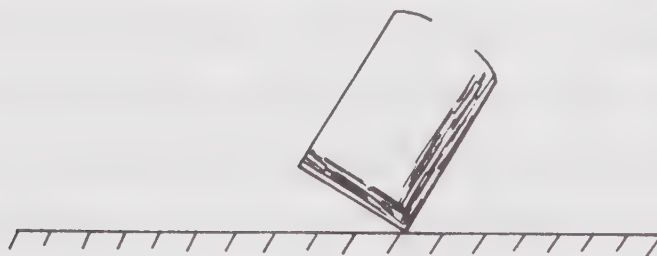


FIGURE 11

ILLUSTRATION SHOWING RESPONSE TO ITEM 5 ON EUCLIDEAN TEST





The next phase in administering this item required subjects to represent the water in the tilted beaker as it was held for them to view. There were 19 incorrect responses on this phase of the test item with 16 of the incorrect responses belonging to the 20 subjects being discussed. All of the 16 responses consisted of a horizontal and a vertical line rather than the correct oblique. All 6 subjects who gave incorrect responses in the previous phase of this item were included in the 16 who gave incorrect responses in this phase.

Responses on item 6 followed a similar pattern with four of the 20 subjects representing the plumb line in a tilted beaker as a horizontal and vertical line as they were asked to anticipate what the plumb line would look like if the beaker were tilted. Twelve of the 20 subjects gave an identical response when shown the plumb line in the tilted beaker.

In comparison, when the beakers were in an upright position only two subjects from the total group of 100 were unable to represent accurately the water and only one subject was unable to give an accurate representation of the plumb line.

This finding appears to support the notion that young children find the representing of oblique lines more difficult than the representing of horizontal and vertical lines. The data indicated that the representing of horizontal and vertical lines were of approximate equivalent difficulty.

Six of the 20 subjects were successful in representing at least one oblique line in the test of Projective Space with only two subjects being successful in representing both oblique lines. This is an indication that there is a relationship between the ability to represent oblique lines in varying situations - transformations in the Euclidean plane and reconstructions which appear to be of a projective



nature.

For each of the four positions in the test of Co-ordination of Perspectives subjects had four sketches from which to choose. The four sketches for each position represented the following viewpoints:

- Subject's view
- Subject's view if the subject leaned in the direction of the 'photographer'
- Reversal of the correct choice
- Correct choice.

Using this criterion meant that the same four choices would not be used for any two positions of the 'photographer'. Because the subject did not change position in arriving at his selection, the sketch to represent his viewpoint was included as one of the four sketches for all four positions of the 'photographer'. Table XXIX shows the number of selections for each sketch in each age category. An explanation of the choices as to which viewpoint each sketch represents is reported in Appendix D. It is difficult to account for some of the findings reported in Table XXIX. The results on the positions of 'right or subject' and 'left of the subject' are almost reversed in nearly all age categories especially when considering the selection of the 'view if leaning toward the 'photographer' and 'Reversal'. When the 'photographer' was placed on the right of the subject 50 percent of the subjects in the age category 85-94 months selected a sketch corresponding to a 'Reversal of the Correct Choice' while 20 percent selected the sketch corresponding to the 'view if leaning toward "photographer"'. When the 'photographer' was placed on the left of the subject 20 percent selected a sketch to correspond to 'Reversal



TABLE XXIX

DISTRIBUTION FOR SELECTIONS OVER AGE CATEGORIES ON TEST OF  
CO-ORDINATION OF PERSPECTIVES

| Age<br>(Months) | Number | Positions of 'photographer'                                                                            |                                                                                                        |                                                                                                        |                                                                                                        |
|-----------------|--------|--------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|
|                 |        | Centre of layout<br>right of subject                                                                   | Left near corner                                                                                       | Opposite subject                                                                                       | Centre layout<br>left of subject                                                                       |
|                 |        | Selections<br>Correct Choice<br>Reversal<br>View leaning<br>toward photo-<br>grapher<br>Subject's View | Selections<br>Correct Choice<br>Reversal<br>View leaning<br>toward photo-<br>grapher<br>Subject's View | Selections<br>Correct Choice<br>Reversal<br>View leaning<br>toward photo-<br>grapher<br>Subject's View | Selections<br>Correct Choice<br>Reversal<br>View leaning<br>toward photo-<br>grapher<br>Subject's View |
| 45-54           | 4      | 1 0 3 0                                                                                                | 1 1 0 2                                                                                                | 0 2 2 0                                                                                                | 0 2 2 0                                                                                                |
| 55-64           | 15     | 2 3 4 6                                                                                                | 4 7 2 2                                                                                                | 5 8 2 0                                                                                                | 5 6 2 2                                                                                                |
| 65-74           | 34     | 5 5 9 15                                                                                               | 9 7 5 13                                                                                               | 6 11 12 5                                                                                              | 9 12 9 4                                                                                               |
| 75-84           | 23     | 0 2 9 12                                                                                               | 3 9 5 6                                                                                                | 4 7 9 3                                                                                                | 5 9 4 5                                                                                                |
| 85-94           | 20     | 0 2 10 8                                                                                               | 1 3 3 13                                                                                               | 2 14 1 3                                                                                               | 4 10 2 4                                                                                               |
| 95-103          | 4      | 0 0 1 3                                                                                                | 0 1 0 3                                                                                                | 2 1 0 1                                                                                                | 0 2 0 2                                                                                                |
| Total           | 100    | 8 12 36 44                                                                                             | 18 28 15 39                                                                                            | 19 43 26 12                                                                                            | 23 41 19 17                                                                                            |



of the Correct Choice' while 50 percent selected a sketch corresponding to 'view if leaning toward "photographer"'. On the previous position there was 40 percent accuracy for the age category 89-94 months whereas only 20 percent of the subjects in this age category gave an accurate response when the 'photographer' was placed in the left position.

It will be recalled that the correct sketch when the 'photographer' was on the left of the subject was significantly more difficult than when the 'photographer' was placed on the right of the subject. It was suggested that this may have been due to the absence of one cone from the left viewpoint. However, this does not seem to account for the vast differences in the choice of sketches representing a 'Reversal of the Correct Choice'. It is also difficult to understand why 52 percent of the subjects in the age category of 75-84 months were successful when the 'photographer' was to the right of the subject but only 40 percent in the age category 85-94 months were successful on this position.

A Kolmogorov-Smirnov Test indicated no significant differences in the distributions for each position except the position on the right of the subject where the distribution between the subject's view and the correct choice is significant at the .05 level. (The required deviation at the .05 level is 0.5227. The deviation obtained was 0.5228.)

The total number of responses for each selection category is shown in Table XXX. The distributions, for combined responses on all four positions between the subject's view and the total correct responses are significantly different at the .01 level (Kolmogorov-





TABLE XXX

COMBINED RESPONSES FOR CORRESPONDING SKETCHES FOR FOUR POSITIONS  
ON TEST OF CO-ORDINATION OF PERSPECTIVES

| Selection                                                       | Age (Months) |    |       |     |       |    |       |    |       |    |
|-----------------------------------------------------------------|--------------|----|-------|-----|-------|----|-------|----|-------|----|
|                                                                 | 45-54        |    | 55-64 |     | 65-74 |    | 75-84 |    | 85-95 |    |
|                                                                 | No.          | %  | No.   | %   | No.   | %  | No.   | %  | No.   | %  |
| Subject's View                                                  | 2            | 13 | 16    | 27  | 29    | 21 | 12    | 13 | 7     | 9  |
| Subject's View<br>if Leaning in<br>Direction of<br>Photographer | 5            | 31 | 24    | 40  | 35    | 26 | 27    | 29 | 29    | 36 |
| Reversal                                                        | 7            | 43 | 10    | 15  | 35    | 25 | 27    | 29 | 16    | 21 |
| Correct<br>Choice                                               | 2            | 13 | 10    | 18  | 37    | 28 | 26    | 29 | 28    | 24 |
| Total                                                           | 16           | 60 | 60    | 136 | 92    | 80 | 16    | 56 | 9     | 16 |



Smirnov Test). The required deviation at the .01 level is 0.2506. The deviation obtained was 0.2537. No other distributions are significantly different from each other.

The subject's tendency to choose the sketch corresponding to his own viewpoint is assumed to represent the characteristic of egocentrism. The significant difference in distributions reported in Table XXX is therefore indicative that there is a greater degree of egocentrism in younger subjects as measured by the Test of Co-ordination of Perspectives.

### Specific Observations

Most of this chapter has dealt with hypotheses testing involving the 100 subjects in Group 1 as a whole. Consequently, any conclusions and generalizations can be applied only to the complete group of subjects. As a means of looking more specifically at individuals it was decided to look at the performance of two individuals in a particular classroom in each of the four 'grades' from which the subjects in this sample were taken. The individuals were so chosen to represent as wide a discrepancy as possible. The subject's scores on each subtest and the total test scores are reported in Table XXXI.

With the exception of the two Grade II subjects, each pair of subjects were in the same classroom. This table suggests that although analysis may indicate developmental trends, in any given classroom there may exist a wide range of abilities.

### Overview of the Findings

The purpose of this section is to investigate relationships which may exist between the responses on the various tests and between



TABLE XXXI  
TEST SCORES FOR SELECTED INDIVIDUALS

| Subject              | Age<br>(Months) | Topological<br>Test | Euclidean<br>Test | Projective<br>Test | Co-ordination<br>Perspective | Total<br>Score |
|----------------------|-----------------|---------------------|-------------------|--------------------|------------------------------|----------------|
| Total Sample<br>Mean | 74.68           | 14.02               | 15.37             | 3.82               | 1.12                         | 34.33          |
| Playschool           | 1 53            | 10                  | 4                 | 1                  | 1                            | 16             |
|                      | 2 57            | 14                  | 15                | 1                  | 0                            | 30             |
| Kindergarten         | 1 70            | 5                   | 2                 | 0                  | 1                            | 8              |
|                      | 2 70            | 15                  | 20                | 6                  | 3                            | 44             |
| Grade I              | 1 78            | 12                  | 14                | 3                  | 2                            | 31             |
|                      | 2 78            | 17                  | 21                | 6                  | 2                            | 46             |
| Grade II             | 1 92            | 16                  | 16                | 3                  | 1                            | 36             |
|                      | 2 90            | 16                  | 21                | 6                  | 2                            | 45             |



the conclusions on the hypotheses which have been tested.

On the basis of the previous analysis and the observed behaviour of the subjects it is apparent that the responses can be divided into two categories - those related to topological aspects and those related to Euclidean aspects. Table XXXII shows the distributions of subjects who were successful on some topological and Euclidean aspects from each of the three tests of Topological, Euclidean and Projective Space. Success on the closure category of Topological Space was taken as two or more out of a total possible of three. On the two items from the test of Euclidean Space success is defined as three or more out of a total possible of four. Success on Problem 1 of the test of Projective Space was a score of 1 since it was scored as 1 or 0 depending on whether or not it was correct.

Table XXXII indicates very close agreement of success on the selected problems on the three tests. The unequal number of subjects in each age category, and in particular the relatively large number of subjects in the category of 65-74 months, favors the median age to be around 70 months. Hence the use of median ages in this case is not as strong an indicator of the relative development of those aspects of spatial abilities as it would have been had all age categories an approximately equal number of subjects. The age of ascension (age at which 50 percent of the subjects were successful) was intended to be used in an attempt to make such a comparison more meaningful. However, as is evident from Table XXXII the percentage of accurate responses made such a procedure inappropriate.





TABLE XXXII

DISTRIBUTION ON SELECTED ITEMS FROM TOTAL TEST OVER AGE CATEGORY

| Age<br>(Months)     | No. | Subject Successful on Topological Aspects |     |                           |     | Subjects Successful on Euclidean Aspects         |     |                                              |     |
|---------------------|-----|-------------------------------------------|-----|---------------------------|-----|--------------------------------------------------|-----|----------------------------------------------|-----|
|                     |     | Closure on Topological Test               |     | Closure on Euclidean Test |     | Line Parallel to Edge. Problem 1 Projective Test |     | Presence of Straight Lines on Euclidean Test |     |
|                     |     | No.                                       | %   | No.                       | %   | No.                                              | %   | No.                                          | %   |
| 45-54               | 4   | 3                                         | 75  | 3                         | 75  | 2                                                | 50  | 3                                            | 75  |
| 55-64               | 15  | 10                                        | 67  | 12                        | 80  | 10                                               | 67  | 11                                           | 73  |
| 65-74               | 34  | 32                                        | 94  | 27                        | 80  | 26                                               | 76  | 30                                           | 88  |
| 75-84               | 23  | 23                                        | 100 | 22                        | 98  | 22                                               | 98  | 23                                           | 100 |
| 85-94               | 20  | 20                                        | 100 | 20                        | 100 | 20                                               | 100 | 19                                           | 95  |
| 95-103              | 4   | 4                                         | 100 | 4                         | 100 | 4                                                | 100 | 4                                            | 100 |
| Median Age (Months) |     | 74.2                                      |     | 75.4                      |     | 76.3                                             |     | 75.0                                         |     |



Table XXXIII reports the distribution of success on non-topological aspects of spatial abilities. The first three items were scored 1 or 0. Hence in those cases success was defined as a score of 1. Success on the fourth item was taken as 3 or more out of a possible score of 4.

Table XXXIII shows a fairly narrow range in both the median ages and the ages of ascension. In all cases the median ages as reported in this table are considerably higher than those reported in Table XXXII. This suggests a general agreement of topological and non-topological properties across the three tests of Topological, Euclidean and Projective Space. It does not, however, indicate the primacy of topological spatial abilities over the abilities encountered in Euclidean and projective space since the ability to represent straight lines, as reported in Table XXXII, seems to develop simultaneously with topological properties. This is a tentative conclusion and will be discussed in more detail in Chapter V.

#### IV. SUMMARY

This chapter contains an evaluation of the instruments used in the study, and the results of testing five hypotheses which were associated with the major purposes of the study as outline in Chapter I. In addition, some general and descriptive analyses are reported.

Two different modes of representation of geometric shapes were compared and it was found that plasticine strips and pencil-paper were not significantly different as media in permitting young children to exhibit the kind of spatial ability investigated in this study.

The main purpose of the study was to make suggestions for



TABLE XXXIII  
DISTRIBUTION OF SELECTED ITEMS FROM TOTAL TEST OVER AGE CATEGORY  
(NON-TOPOLOGICAL ASPECTS)

| Age<br>(Months)  | No. | Horizontal<br>Representation<br>Item 5 |    | Vertical<br>Representation<br>Item 6 |    | Oblique Line<br>Problem 3 |    | Accuracy of Lines<br>Euclidean Test |    |
|------------------|-----|----------------------------------------|----|--------------------------------------|----|---------------------------|----|-------------------------------------|----|
|                  |     | No.                                    | %  | No.                                  | %  | No.                       | %  | No.                                 | %  |
| 45-54            | 4   | 0                                      | 0  | 0                                    | 0  | 0                         | 0  | 0                                   | 0  |
| 55-64            | 15  | 1                                      | 7  | 1                                    | 7  | 2                         | 13 | 1                                   | 7  |
| 65-74            | 34  | 7                                      | 21 | 5                                    | 15 | 8                         | 24 | 11                                  | 29 |
| 75-84            | 23  | 10                                     | 44 | 14                                   | 61 | 14                        | 61 | 11                                  | 48 |
| 85-94            | 20  | 11                                     | 55 | 14                                   | 70 | 13                        | 65 | 13                                  | 65 |
| 95-113           | 4   | 3                                      | 75 | 3                                    | 75 | 3                         | 75 | 3                                   | 75 |
| Median Age       |     | 82.5                                   |    | 83.8                                 |    | 81.6                      |    | 82.7                                |    |
| Age of Ascension |     | 85.6                                   |    | 80.2                                 |    | 78.8                      |    | 84.8                                |    |



geometry activities for elementary grade children. Associated with this purpose was the secondary purpose of investigating the development of spatial abilities in young children. This involved the analyzing of subjects' responses on four tests concerned with Topological, Euclidean and Projective Space and the Co-ordination of Perspectives. A general analysis indicated that age was significantly related to the development of the abilities investigated by those tests.

Further analysis indicated that the topological aspect of ordering on a 'figure-eight' shaped configuration was significantly more difficult than the other aspects of topological space which were investigated; namely closure, separation, proximity and ordering beads on a straight wire to model identical beads on a circular shaped wire.

On the test of Euclidean Space the mean scores on items demanding accuracy of lines and accuracy of angles were significantly different from the mean scores on items requiring the subjects to indicate the presence of lines and angles. On the basis of this finding regarding category difficulty a developmental scale was devised. The subjects were divided into six age categories and distributed over four divisions. There was a significant difference between the distributions of those subjects who were successful only on the division requiring success on the presence of lines and angles and those subjects in the division requiring accuracy of lines and angles. It was also found that the distribution of subjects in the division requiring success on the horizontal and vertical categories was significantly different from those subjects in the division requiring accuracy of lines and





angles. In summary, the Euclidean properties appear to be ordered with age as follows: presence of lines and angles, accuracy of lines and angles, success on items involving horizontal and vertical transformations in the Euclidean plane; that is, items 5 and 6 on the Euclidean Test.

The mean scores on problems involving oblique lines on the test of Projective Space were significantly lower than the scores on the other problems involving the construction of lines near the edge of the base and central straight lines. The distributions on a scale which considered this difference in difficulty indicated that older subjects were much more successful on constructing oblique lines than were younger subjects.

On the test of Co-ordination of Perspectives the mean scores on the two positions where the sketches corresponding to the correct response contained one less object than the sketch representing the subject's view were significantly different from the mean scores on the other two positions. For those two positions the correct sketches each contained the same number of objects in the subject's view. This finding was taken into consideration when a scale was set up to investigate the distributions over age categories. There was no indication, however, that older subjects did significantly better on the more difficult problems. This test was further investigated by considering the incorrect responses of the subjects. It was found in this investigation that when the responses were totalled for all four positions there was a significant difference in the distributions over age categories between the selection of correct responses and the selection of those responses corresponding to the subject's viewpoint.



There was general agreement in the age of development of the topological aspects of spatial abilities across the three tests of Topological, Euclidean and Projective Space. A similar consistency was found when non-topological properties were compared across the two tests of Euclidean and Projective Space. However, the ability to represent straight lines and angles, without attention to accuracy, appeared no more difficult than the topological properties of closure, proximity and separation.

The responses of selected individuals dealing with oblique lines were consistent across the two tests of Euclidean and Projective Space.

Implications arising from these findings will be discussed in Chapter VI.



## CHAPTER V

### SUMMARY, DISCUSSION AND RECOMMENDATIONS

#### I. SUMMARY OF THE INVESTIGATION

The present study was designed primarily to investigate some aspects of spatial abilities of young children and on the basis of those findings to suggest some suitable kinds of mathematically related activities for the elementary grades. Attention was focused on Topological, Euclidean and Projective Space and the Co-ordination of Perspectives. Those aspects of spatial abilities were chosen to permit investigation into topological spatial abilities in the Euclidean plane. Emphasis was placed on the apparent complexities of the various aspects of spatial abilities and their order of development in young children. The findings were used as guidelines for the inclusion and sequencing of the various activities which have been suggested in the next chapter.

In order to gather the necessary data a sample of playschool, kindergarten, Grade I and Grade II children was selected and the various tests of spatial abilities were employed.

#### Sample

The sample of one hundred and ninety-five children was chosen by randomly selecting from a total population of eight hundred and fifty children in playschool, kindergarten, Grade I and Grade II from schools administered by the Edmonton Public School Board. The ages of these children ranged from 47 months to 103 months with a mean age of 74.62 months.



## Instruments

The total sample was divided into two groups of one hundred subjects and ninety-five subjects respectively. Group 1, one hundred subjects, was administered four tests of spatial abilities. The test of Topological Space contained five items and was concerned with properties of closure, proximity, separation and order. The test of Euclidean Space contained six items and investigated closure, the abilities to represent angles and lines in the Euclidean plane, and the notions of horizontal and vertical representations. The ability to represent straight and oblique lines on a rectangular and a circular base constituted the six items on the test of Projective Space. The test of Co-ordination of Perspectives contained four items in an attempt to investigate the integration of the concepts contained in the three previous tests. There was a total possible score of forty-nine on the total test of spatial abilities. The reliability as calculated by the Kuder Richardson Formula 20 was 0.796.

Items 1, 2 and 3 from the test of Topological Space and items 1, 2, 3 and 4 from the test of Euclidean Space constituted the test which was administered to the ninety-five subjects in Group 2. These subjects represented shapes by using strips of plasticine; in Group 1, subjects represented the same shapes by using pencil and paper. This test was included to permit a comparison between the different modes in representing two-dimensional geometrical shapes. For this test the reliability was 0.7241 (Kuder Richardson Formula 20).

## II. MAJOR FINDINGS

The summary of the findings discussed in this section will mainly be in terms of the hypotheses which were tested. The next





section of this chapter will include a more subjective discussion of the findings.

It was found that there were no significant differences on the criterion scores when the responses of the group of subjects who represented the two-dimensional shapes by pencil-paper were compared with the responses by subjects who represented the same shapes by using strips of plasticine. A general analysis indicated that age was significantly related to all aspects of spatial abilities investigated in this study. This latter finding formed the basis of much of the detailed analysis reported in testing the hypotheses.

Hypothesis One investigated responses on the total test of spatial abilities with regard to sex differences. It was found on each of the tests of Topological, Euclidean and Projective Space and the test of Co-ordination of Perspectives that the performances of girls and boys were not significantly different.

Hypothesis Two dealt with category difficulty and distribution over age categories on the test of Topological Space. It was found that the means on categories of closure, separation and proximity were not significantly different from each other. The mean scores on the two categories dealing with order were significantly different from each other.

The developmental scale which was formed on the basis of the category difficulty distributed the one hundred subjects in Group 1 over four divisions and six age categories. The analysis indicated that the distribution of subjects who were successful on both ordering items was significantly different from the distribution of subjects who were unsuccessful on the 'ordering' aspect but successful on the topological



aspects of closure, proximity and separation. That is, older subjects responded with a greater degree of accuracy than did the younger subjects of the sample.

Hypothesis Three investigated category difficulty and distributions on the test of Euclidean Space. The mean scores on the categories of closure, presence of angles and presence of straight lines were not significantly different from the mean scores on the categories of accuracy of lines and accuracy of angles. The closure property on this test was scored as a purely topological aspect. Hence, the findings on this hypothesis did not indicate that the topological property of closure was significantly easier for children to attain than was the Euclidean properties of representing lines and angles.

The distributions on accuracy of lines and angles were also significantly different from the distribution on categories of closure, presence of angles and presence of straight lines. This indicated that a greater proportion of the older subjects was successful on the categories of the test of Euclidean Space which demanded accuracy in the representations.

The difficulty of the categories on horizontal and vertical representation were not significantly different from each other.

The construction of horizontal and oblique lines was dealt with in Hypothesis Four. The two problems dealing with oblique constructions were significantly more difficult than those problems requiring the construction of lines near the edge of the bases and those involving central straight lines. The problems were grouped in three divisions to include lines near the edge of the bases, central



straight lines and oblique lines. The subjects' responses were distributed in those three divisions over six age categories. The distributions on the first two divisions were not significantly different from each other but both distributions were significantly different from the distribution on the division of oblique lines. The older subjects showed a high degree of success in all problems, whereas few of the younger subjects gave accurate responses to problems involving the construction of oblique lines.

Hypothesis Five involved investigation into the Co-ordination of Perspectives. This test required subjects to select from four sketches of a 'mountain scene' the choice which correctly identified the view of a 'photographer' who occupied a succession of four different positions. The correct sketches for two of the positions contained two mountains whereas the correct sketches for the other positions and the subject's view of the scene always contained three. Those latter positions were found to be significantly easier than the previous two. The distributions on the four positions, when considered over six age categories, were not significantly different from each other. Therefore, part (a) of Hypothesis Five, dealing with position difficulty was rejected. Subjects apparently experienced great difficulty in imagining a view of the 'mountain scene' which contained fewer 'mountains' than was visible from their own viewpoint.

In summarizing the findings from the various tests it was found that generally the responses were indicative of two levels of conceptualization - topological and Euclidean (or Projective). It is not obvious from the findings in this study, however, that the latter develops later in children than the aspects of topological spatial



abilities. This will be discussed in the next section of this chapter.

Selected responses from each 'grade' level of the sample indicated that there was a wide range in the development of spatial abilities within a given single classroom. There is some suggestion that this range of differences may decrease with the increase in the years of schooling. However, a definite conclusion in this regard is beyond the scope of the present study.

### III. DISCUSSION OF THE FINDINGS

The relationship of age to the various aspects of spatial abilities as investigated in the present study was indicated by correlations. Beyond this general overview it was decided not to employ correlations as a means of analyzing the responses. This decision was made for two reasons. The first centers around the age range in the investigation of any developmental phenomenon. The solution to overcome this difficulty was not obvious. To use partial correlations to control for age may have resulted in partialling out the developmental factor itself - the very aspect which was being investigated. The second reason is implicit in the first. The only variables, other than the variables in the instruments, were sex and age. The responses of the different sexes, consistent with most research in this area, were not significantly different. Hence, age was the only variable remaining. This situation existed because of the difficulty in obtaining information on variables such as intelligence quotients, father's occupation, and previous schooling - variables which have been suggested by research as being influential in child







development. If sufficient information had been obtained, then perhaps correlations and partial correlations would have been justifiably more widely used.

The position taken, therefore, in the present study was to examine the protocols in detail in relationship to the age of the subjects as a means of investigating the developmental trend suggested by the correlations.

The categories of closure, proximity, and separation on the test of Topological Space were of approximate equivalent difficulty. This does not mean that generally it can be concluded that the concepts of proximity and separation develop simultaneously with the concept of closure. This conclusion is not possible from the information in this study since it appears that the shapes to be represented did not provide sufficient opportunities to distinguish those concepts. To make the shapes more complex would have forced consideration and distinction of motor integration and perceptual development. The shapes used were rather primitive but important since they were related to mathematics and they did permit an examination of the closure property.

The category on ordering beads from the circular shaped wire presented little difficulty to most subjects. However, the last category on the test of Topological Space which required subjects to arrange beads on a straight wire to correspond to the order on a 'figure-eight' shaped configuration presented much more difficulty. Almost all the subjects erred at the intersection of the wire. It may be that the subjects forgot that the wires were not joined, although this appears unlikely for two reasons. First, success on



this problem correlated highly with age. (It may be, however, that a memory test would also have correlated significantly with age.)

Secondly, during the training session subjects were shown repeatedly that the wire forming the 'figure-eight' could indeed be made straight and without exception all subjects seemed to understand this part of the instructions. It may be that the subjects did not clearly understand the task but there was no overt evidence during the testing period.

Many subjects who were successful on the categories on the Topological Test were unsuccessful on other aspects of the spatial abilities investigated in this study. Nevertheless, this does not mean that the cognitive structures of those subjects were quite undeveloped. There is evidence (Laurendeau and Pinard, 1970; Piaget, 1967) that the responses of subjects younger than those in the sample in the present study would have been indicative of a more chaotic stage of cognitive functioning. The success of subjects on this test then, indicates the child's systematic attempt to represent a situation by assimilating it to what he already knows. However primitive this system may appear it nevertheless is indicative of a mental organization exhibiting ability to assimilate reality to action.

It was apparent from the subjects' comments as they drew the shapes in both the Topological and Euclidean Tests that many of them realized they had not accurately represented the shape but they appeared quite unable to correct it. For some subjects the inaccuracy was apparent to them in the process of drawing and for others it became evident only when they compared the shapes on completion. Most difficulty was encountered in cases involving oblique lines as in the



triangle and the rhombus. It was also observed that subjects experienced less difficulty in representing the base line of the triangle and the two base angles than they did in representing the apex of the triangle which, of course, involved the intersection of two oblique lines.

Several of the younger subjects inquired of the investigator "Does it look like I'm making one?". When they had completed the drawing the investigator asked if what they had drawn was like the shape presented to them and although in many cases their drawings were mere scribbles those subjects said their shapes indeed resembled the models. It seems unlikely that the younger subjects experienced a purely visual discrimination problem. The difficulty may have been caused by a loss in discriminatory ability in the shift from one model to the other, or it may have been that those subjects really did not believe in the correspondence of the shapes but failed to appreciate the need for the accuracy demanded by the test items. In older subjects, however, even though inaccuracies occurred the shapes were recognized by the subjects as not being accurate representations.

Most of the shapes in both the Topological and Euclidean Tests were easily labelled. An **unusual** number of subjects thought the shape which constituted item one on the test of Topological Space was very difficult. Subjects who labelled the shape a 'foot' usually drew it as such and were generally more successful than those who failed to put a name on the shape.

During the testing period it was observed that the number of eye movements from model to drawing varied greatly with subjects. There was no accurate count of these eye movements but it was apparent that the number increased with older subjects. In Group 2, where





subjects represented shapes by using strips of plasticine, a count was made of the number of strips of plasticine used by each subject to form each shape but this number was not significantly related to any other variable in the test.

The responses to phase two of items 5 and 6 of the test of Euclidean Space required subjects to draw oblique lines. This was found to be much more difficult than phase one of the same problem which required a horizontal and vertical line respectively. A great number of the subjects who had difficulty on those items also did poorly on representing the triangle and the rhombus. There were cases here where subjects realized they had not represented the water or the plumb line accurately but were unable to draw the necessary oblique. A number of subjects resorted to a combination of a horizontal and vertical line as a substitute for a single oblique.

The difficulty for subjects to represent an oblique line in the Euclidean plane was also reflected in three-dimensional space. In the test of Projective Space, which dealt with the projective straight line, the items requiring subjects to construct oblique lines by using matchsticks in clay bases were significantly more difficult than the items requiring straight lines near the edge of the base and central straight lines. It was assumed by Laurendeau and Pinard (1970) that the ability to represent lines other than obliques was governed by perceptual influences offered by the visible contours of the bases on which the lines were formed. This conclusion was suggested by the fact that subjects when required to construct straight lines near the edge of both the rectangular and circular bases were generally successful. However, it must be noted that in the Laurendeau and Pinard study





the circular base was placed on a rectangular table with the segment containing the path of the line to be constructed protruding over the edge. Consequently, it is likely that subjects' responses were greatly influenced by the outline of the table. In the present study both bases were placed on an unpatterned floor and the problem which required a straight line to be constructed near the edge of the circular base (Appendix C, Problem 4) appeared quite difficult for some subjects. The responses were obviously influenced by perceptual cues since many of the subjects constructed lines which followed the arc of the circular base.

Laurendeau and Pinard (1970) also observed that a response to one item on the test of Projective Space greatly resembled the previous response despite the different criterion. They attributed this to invariant perceptual cues from one problem to the next. In the present study, however, it was observed on items 5 and 6 of the Euclidean Test (Appendix C) that subjects sometimes drew a vertical plumb line instead of a horizontal line to represent water and vice-versa depending on the order in which the items were administered. In this case the perceptual field was vastly different from one item to the next because of the introduction of different stimulus. It is not clear why this sometimes resulted and an investigation of this aspect is beyond the scope of this study. It may be that such occurrences are more related to individual learning styles and flexibility of individuals than perceptual influences caused by similarities in the spatial field.

From the findings of the present study it is apparent that success on the construction of straight lines near the edge of the



rectangular and circular bases are greatly influenced by perceptual cues but it must also be noted that those constructions demand attention to the more specific relations of continuity and spatial order which constitute the basis of the topological line. Construction of such lines do not, however, require an internal reconstruction of a real activity as in the construction of the obliques. In mastering the obliques the subject must be able to co-ordinate his movements in the Euclidean plane. The distinction between the two types of problems is a shift from the intuitive to the operational level of functioning which may or may not be manifested by an 'aiming' behaviour involving both the subject himself and the two 'houses' which were placed on the base. Twenty-four of the fifty-five subjects who were successful on the obliques showed no indication that they were 'aiming' in constructing the line. Many subjects assumed a position of looking down on the base and formed the straight line by placing the matchsticks between their hands and gently moving them in a straight line position. This behaviour may have indicated Euclidean spatial abilities with attention given only to the position of the matchsticks relative to the adjacent sides of the rectangular base and the arc of the circular base. Alternatively, in representing the two-dimensional shapes subjects may have made use of projective spatial abilities. (It was not obvious that this was the case.) In summary, it was not possible in the present study to distinguish between those spatial abilities which may be termed Euclidean and those which may be termed Projective. Consequently no comment can be made regarding whether or not the two abilities develop simultaneously. Laurendeau and Pinard (1970) in their study on the development of the concept



of space in the child reached a similar conclusion.

The test of Co-ordination of Perspectives was administered to investigate the integration of topological and Euclidean spatial abilities. In administering this test a number of difficulties were encountered and various inconsistencies became evident. The testing areas in some of the schools in which data was collected were too small to isolate the testing apparatus. This prevented subjects from being able to move around the layout to verify their responses. If the testing areas had been sufficiently large a more intensive training period would have been undertaken. This would have provided increased assurance that the subjects understood exactly what was required of them.

Various inconsistencies in the responses on this test were discussed in Chapter IV and are verified in Table XXIX. These inconsistencies are difficult to account for. Improved results may be obtained in future administering of this test if a testing area of ample size is used and an intensive training period is undertaken. It may be however, that the whole situation is too complex for such young children as was used in this study.

There are two main conclusions to be drawn from the results of the test in this study. First, the most difficult views for subjects to imagine were those containing one object less than the number of objects present in their viewpoint, and secondly, there is an indication of an egocentric attitude being more predominate in younger subjects than in older subjects. This conclusion was arrived at by an investigation of the incorrect responses. Laurendeau and Pinard (1970) reported similar findings.



The findings of the present study do not support the theory (Piaget, 1967) that topological abilities develop in children earlier than spatial abilities of a Euclidean nature. This was evident in that the closure category of the test of Euclidean Space, which was scored from a topological viewpoint, was of approximate equivalent difficulty as the categories requiring subjects to represent lines and angles. It was also found that the median age of success on the topological categories of closure, proximity and separation was very close to the median age on the Euclidean categories of presence of angles and presence of lines. This result is not surprising in view of an analysis of the problem situations involved. It appears unlikely that there is a great deal of co-ordination or organization necessary within the Euclidean plane to ensure success on the categories of presence of lines and angles. Subjects must be aware of the continuity and order in representing straight lines which is basically a topological characteristic. In representing angles the subject must be able to co-ordinate the forming of one line with the preceding line which he has represented. If this is represented in a very gross manner the response is still indicative of the subject's ability to represent angles. It is not necessary in the solution to such a problem for subjects to be aware of the relative location of the model in the Euclidean plane, rotation is not important and neither is the size of the angles nor the lengths of the lines. If those things were considered and if the requirement were an accurate representation of two-dimensional shapes, then the problem becomes much more difficult and the ability required for success on problems of this nature appears to occur later in children than the ability to indicate the mere







presence of lines and angles. That is, the findings of this study indicate that it is the accuracy in representing two-dimensional shapes in the Euclidean plane that develops later than the more intuitive abilities associated with topological representation. It appears that this is the case because of the attention which the subject must give to the spatial field both within and external to the model itself.

On the items concerning the projective straight lines perceptual cues had an obvious influence on the responses especially of younger subjects and obliques demanded co-ordination and organization beyond the ability of many subjects. The difficulty in analyzing this test was whether the construction of an oblique is indeed indicative of projective abilities or whether the ability to operate in a Euclidean plane is sufficient.

#### IV. RECOMMENDATIONS FOR FURTHER RESEARCH

This study was concerned with an investigation and related activities on only part of the total area of spatial abilities. One recommendation for further research is a study into other aspects of spatial abilities of young children such as tactile perception and exercises involving transformations. This would provide a broader base on which to formulate a more complete mathematics program in the elementary grades.

A second recommendation for further research is related to information on subjects. It is felt that information on variables such as language comprehension, intelligence quotients, previous schooling and home background would be valuable in permitting some generalizations regarding the development of spatial abilities. It is



suggested that if such a study were undertaken the sample should be limited, since obtaining data and the detailed processing of data of this kind is very time consuming and difficult.

A third recommendation for further research in the elementary grades is of the kind presently being undertaken by Bober (1972) with high school students. This research is a pretest - training - post-test design. The pretest and post-tests might be tests of spatial abilities similar to the instruments used in this study. The activities might be of the kind suggested in Chapter VI.

It was considered impossible in this study to determine the abilities being exhibited in the responses to some items on the test of Projective Space. Related to this problem is the presence of what Piaget terms horizontal décalages. A fourth recommendation for further research concerns the analysis of subjects' responses. The recommendation is for a study in the investigation of general spatial abilities and the responses factor analyzed to determine the factors involved in the responses without predetermined labels on the kinds of spatial abilities being tested. It may be that many problems demand kinds of abilities quite different from those suggested by theoretical considerations.



## CHAPTER VI

### IMPLICATIONS AND ACTIVITIES BASED ON THE FINDINGS OF THE STUDY

The ultimate purpose of this study was to provide suggestions for activities which could be used in elementary grade classrooms. The activities which have been suggested have relevance to mathematics as a discipline. However, the prime motivation for those suggested activities arose from the findings regarding the spatial abilities of young children. Teachers and mathematics educators recognize the need for activities in elementary grade mathematics which will aid in the development of spatial abilities, promote a higher level of cognitive functioning and generally aid the child in establishing some organization to what otherwise might be chaotic surroundings.

In the previous chapter the findings of the investigation into some aspects of spatial abilities of young children were reported without reference to implications for elementary mathematics. In this chapter the findings of the study are summarized with specific suggestions relating to the development of part of a mathematics program for young children. The suggestions must be interpreted in view of the limitations of the study.

The findings suggest the following:

- For any given age the spatial ability of boys and girls are not significantly different.
- Spatial abilities increase significantly with age.
- The topological abilities of closure, proximity, separation and simple ordering of objects are well developed in most children by the time they enter kindergarten (around 5 years). Children of this age and older enjoy activities



involving those aspects but a program of readiness appears to be unnecessary in this area.

- Ordering from a shape such as a 'figure-eight' configuration presents difficulty for many children in playschool, kindergarten, Grade I and Grade II.
- Most children from 5 years possess the ability to recognize and represent straight lines and angles. This appears to be no more difficult than the representation of the topological aspects of closure, proximity and separation.
- There is a significant development with age in the ability to represent straight lines and angles with accuracy. Whereas, it appears unnecessary to implement any program with activities leading up to recognizing and representing lines and angles, the findings do imply that activities of this nature are necessary to develop accuracy in those areas.
- Children in those earlier years (5-8 years) have great difficulty in sufficiently structuring their thought to overpower the perceptual cues of the spatial field. This is much more pronounced in younger children (5 years) than in older children (8 years). In the present study this was evident in the items on horizontal and vertical transformations and on the construction of the projective straight line. The suggestion is, therefore, to engage children in sequenced activities which require children to take an increasing number of perceptual cues into consideration in solving a problem. That is, depending on the 'conflict' with which a given child is able to cope, the perceptual cues could be strengthened or increased in number.
- The construction of an oblique line is almost impossible for most 5 year olds and presents quite some difficulty for 8 year olds. This difficulty appears related to location and orientation in the Euclidean plane. Activities involving drawing in two dimensions should be included in elementary grades.
- It is difficult for young children (5-8 years) to imagine a setting from a viewpoint other than their own. It reflects perhaps a characteristic of egocentrism which appears prevalent in such young children. Because of the difficulty which most children in the elementary grades seem to experience in this aspect of spatial abilities, activities should be in a simple setting. Exercises which are highly contrived and embedded in complex settings should perhaps be delayed until latter grades.

The activities and suggestions contained in this chapter are not to be regarded as representative of a complete mathematics







program for elementary grades. They are representative only of the kinds of activities associated with the aspects of spatial abilities which were investigated in this study. Hence, there are many areas of mathematics which may be suitable for elementary grades and which should be included in a total elementary program which will receive no mention in the activities which follow.

In considering the activities it must also be remembered that they have been tried very minimally with elementary grade children. They are presented as a guide rather than a course of instruction and, therefore, it is expected that modification is necessary in many cases. In devising the activities the findings of the investigation was a basic consideration. An attempt was also made to try to make the activities simple enough to be attractive but challenging enough to be interesting. Activities for group work were considered as well as opportunities for independent investigations. The materials which have been suggested are generally accessible and are used to relate to some mathematical concept rather than the scientific behaviour of the transformed objects which may be involved. That is, an effort has been made to focus the subject's attention on properties arising from action on the **objects** rather than on the property of the objects themselves.

The following activities are suggested for elementary grade children in the age range of 5-8 years although it is recognized that many children outside this age range may find some of the activities challenging and rewarding.



## ACTIVITY 1

Purpose: To provide experiences to assist children in recognizing the characteristics of curves, lines, and angles in 2-D and 3-D shapes.

Materials: A supply of geometric solids of different shapes and colors.

Procedure: Children are asked to put the geometric solids into different groups. The number of groups will depend on the attributes which the child is using. If, for example, the shapes are only two colors, he may decide to have one group for each color. This may be an interesting and challenging activity for some children.

Other children will group the objects according to their shapes. A gross classification may be those that roll easily and those that do not roll. The different groups will depend on the child and the shapes provided. In selecting objects care should be taken to have similar shapes of different colors and dimensions; for example, have several cylinders of different colors and sizes. This gives the child an opportunity not only to compare differences in shapes like cylinders and rectangular solids but also an opportunity to compare shapes for identical characteristics. The variety of shapes will depend on the capabilities of the children involved in the exercise.

A similar activity could involve classifying two dimensional shapes cut from colored paper on bristol board.

Such an activity should help to make children aware of the characteristics which distinguish geometric shapes.



## ACTIVITY 2

Purpose: To provide experiences regarding closure, continuity, separation and proximity.

Materials: Sheets of thin rubber approximately 6" by 6".  
Felt markers.

Procedure: As an orientation to the planned activity suggest that children draw any shapes they wish on the rubber sheets. Stretch the rubber in various directions and watch how the shapes change. (Many children will respond readily to the situation without any suggestions by the teacher.)

Following this introduction to the materials the teacher could draw a circle on a rubber sheet and ask the pupils to draw a similar shape. Some pupil's drawings will be rather crude but almost all pupils at this level will draw a closed curve which will resemble a circle. Experimenting with this shape will convince children that the circle will change shape on stretching but will remain a closed curve. Next, a mark to represent a boy or girl is placed outside the circle and another mark to represent a rabbit is placed inside. The object of stretching the rubber sheet is to see if the distance between the boy (girl) and the rabbit changes and if the rubber can be stretched so that the boy (girl) is inside the circle or the rabbit is outside.

The activity could be continued by changing the shapes to be represented on the rubber sheet (shapes should be kept simple) and children could suggest shapes and invent situations to be investigated.

Guidance from the teacher should result in the children concluding that the topology of a representation is unchanged by the transformations. It is not suggested, of course, that pupils or



teachers would use this terminology but situations leading to this conclusion might include the following:

- an arc of a circle being made straight
- a circle stretched but not made straight and not an open curve
- an open curve not becoming a closed curve
- two lines which intersect not becoming disjoint and vice versa
- path drawn from point A to point B changing in length but remaining unbroken.

### ACTIVITY 3

Purpose: To create an awareness of boundaries (closure).

Materials: Large plastic hoops some of which have been cut to form open curves.

Procedure: Place the open and closed hoops on the floor and place objects inside and outside the closed hoops and in the corresponding positions for the open hoops. Children could best be used as those objects. (Play a game resembling the 'Simon Says' game.) The teacher or a pupil could give the instructions which would require the children to move inside the hoops or outside and hence would stress the notion of boundaries. A pupil is out of the game when he makes a move contrary to the instructions.

A discussion could take place regarding the similarities and differences of the closed and open hoops. Children could be asked to find examples of open and closed shapes in their classroom, an example of which might be the room itself with the door closed and another with the door open.





ACTIVITY 4

A shortage of material for mathematics can sometimes be compensated for by pupils using their bodies to perform the desired transformations.

Purpose: To create an awareness of words and concepts related to proximity, separation and enclosure.

Materials: None necessary.

Procedure: The teacher could involve all the children in activities in 'body' mathematics. No classroom reorganization is necessary. Children can stand by their desks. The teacher may call the instructions at first but it is suggested that this role be taken in turn by pupils as soon as the game is familiar to them.

The instructions might include the following:

- place your left hand near your chin
- put your right hand far from your face
- join hands with a partner to make a shape with no corners
- join hands with a partner to make a shape with two corners
- sit on the floor, make yourself into a shape that will roll
- lie on the floor. Roll to the left, roll to the right
- turn so that your head will be where your feet are now.

A pupil retires from the game when he realized that he has not followed instructions. He demonstrates to the class what he did wrong and what he thinks the right procedure would have been. He is then permitted to continue the game with his classmates. The teacher, even though she may see errors, does not ask pupils to leave the game.



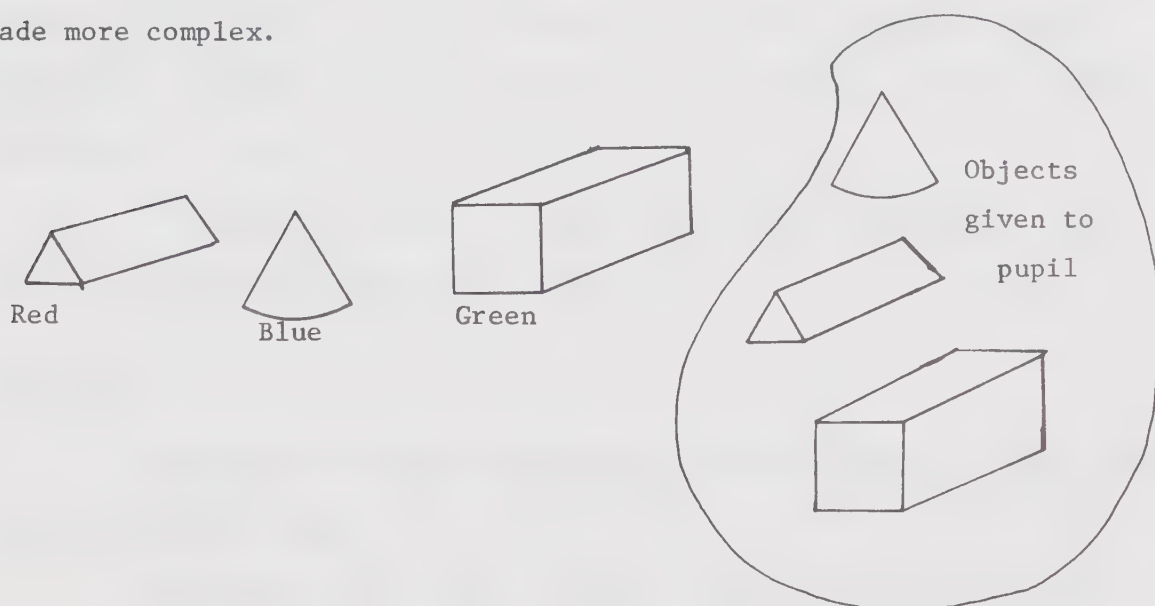
## ACTIVITY 5

Purpose: To provide experiences in the topological property of ordering and the Euclidean aspect of location in the Euclidean plane

Materials: Seven or eight objects of different shapes and colors.

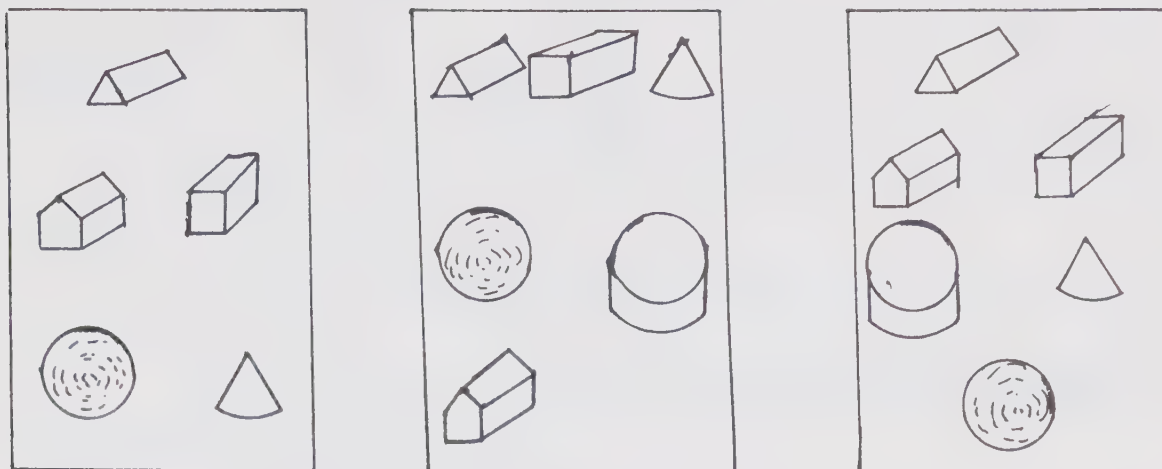
Procedure: Arrange three objects in a straight line and provide the pupil with three similar objects and ask him to arrange his objects to look like those in the straight line. An example is shown below.

The difficulty encountered by the pupil will indicate the rate at which the number of objects should be increased or the patterns made more complex.



If the straight line ordering appears too simple arrange objects in a circle or some other pattern. Some patterns are suggested below.





A development of the concept of ordering would imply that it is best to move from objects differing by more than one attribute to objects differing by only one attribute. The activity can be varied by requiring pupils to copy the order of objects which have been strung on a wire or stick.

If geometric solids or beads are not available objects such as books or colored paper can be used.

#### ACTIVITY 6

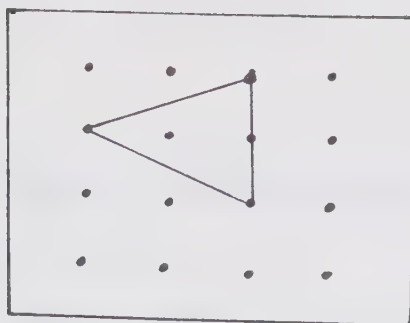
Purpose: To provide opportunities to represent two dimensional open and closed curves.

Materials. Geoboards, supply of rubber bands and assignment-cards containing shapes and patterns to be copied.

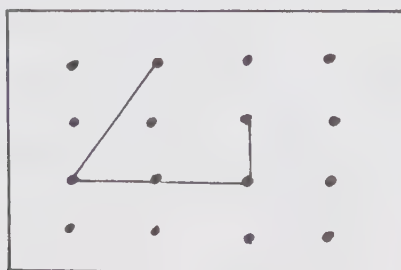
Procedure: Depending on the quantity of geoboards children could work independently or in small groups.

The first part of the activity requires the child to copy on a geoboard a closed shape from an assignment card which is provided. A sample card is shown.





After a number of such cards the shapes could be changed to simple open shapes as shown.



Next children could be required to make closed shapes from open shapes on the geoboards. Appropriate assignment cards could be provided.

Children could also be given instructions orally and be required to make such shapes as follows:

- closed shape with 4 sides
- closed shape with 3 sides
- try to make a closed shape with 2 sides
- make an open shape with 2 sides.

More advanced children might like to make patterns by expanding on a given pattern. An example is shown.







ACTIVITY 7

Purpose: To create an awareness of the relative characteristics of two dimensional shapes.

Materials: Sheets of plain paper or cardboard and a supply of cardboard shapes - circles (approximately  $1\frac{1}{2}$ " in diameter), squares (approximately  $1\frac{1}{2}$ " sides), rectangles (approximately 1" by 2"), and triangles (approximately  $1\frac{1}{2}$ " sides).

Procedure: Each pupil requires a plain sheet of paper or cardboard and a sufficient number of shapes to cover it completely. The object of the activity is to use each group of shapes to try to completely cover the sheet of paper leaving no places where the paper can be seen through the shapes being used but it is permitted for the shapes to protrude over the edge of the paper.

Get pupils to sort the supply of shapes into four groups, each group corresponding to the four shapes to be used. First get them to try covering the sheet with squares. When they have been successful with this shape they might try using the rectangles or triangles. When the circles are used the teacher should initiate a discussion on the characteristics of the shapes and discuss why the circles will not work in this problem.

In this way children will become aware of the relative characteristics of two dimensional shapes. Attention could be directed to objects in the classroom or objects which the pupils can think of which might be used to cover the plane surface and those which could not be used.

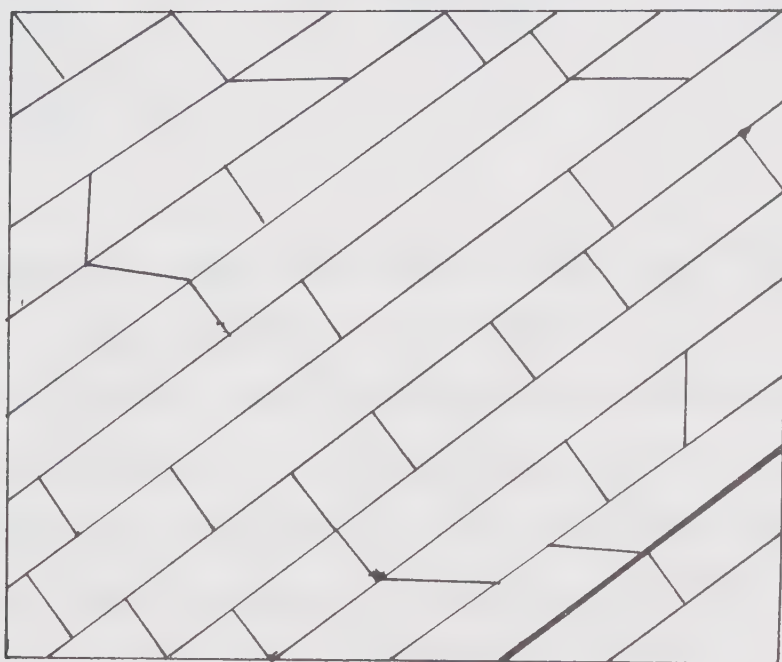


ACTIVITY 8

Purpose: Exercise in constructing and recognizing two-dimensional shapes.

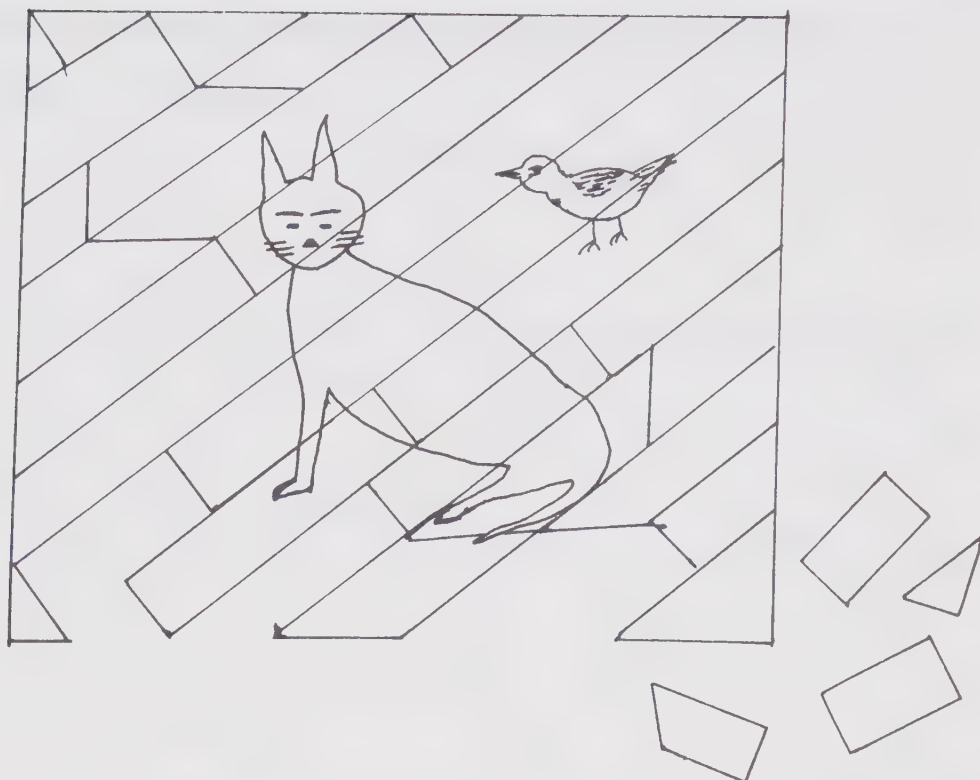
Materials: Cardboard or some similar material from which to make jigsaw puzzles. Paper cutter or scissors.

Procedure: Pupils are supplied with sheets of cardboard or some similar material which have been divided by lines. An example is shown.



They can work independently or in pairs to draw a picture of their choice on the sheets which have been provided and then cut along the lines to form individual pieces.





Children can then assemble the puzzle they have made or they might exchange and assemble puzzles of other pupils.

As an added activity the pieces from a number of puzzles could be mixed together. The assembling of a number of puzzles mixed in this way provides a challenge for the more advanced pupils. Other pupils might find it interesting and challenging to make a puzzle by drawing their own lines as well as their own pictures.

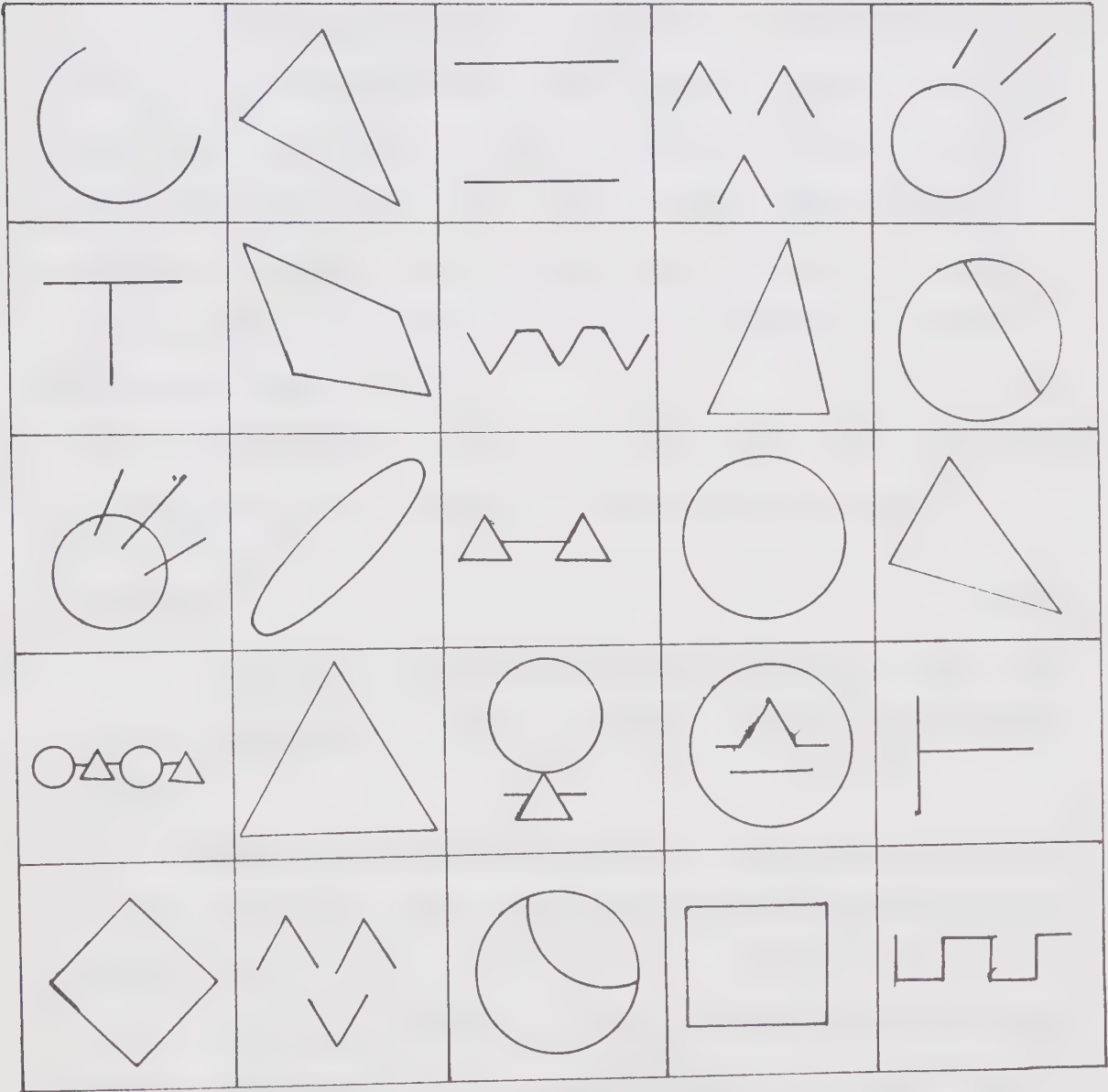
### ACTIVITY 9

Purpose: To provide practice in recognizing two-dimensional shapes which involves discriminating sizes of angles and relative lengths of sides, as well as recognizing positions and patterns.

Materials: A number of small plastic discs or some similar object, a supply of paper approximately 8" by 8" divided into sixteen



or twenty-five squares. Each square contains a shape as in the illustration below. A supply of small square cards containing shapes to resemble those on the sheets.







Procedure: All sheets may be alike. Different arrays of shapes can also be interesting. The game is played much like 'Bingo'. The teacher or one of the pupils holds up a small square containing a shape and the player places a disc over that shape if he can locate it on his card. The first person to form a horizontal, vertical or diagonal row with the discs is declared winner of the game.

If the teacher does not have access to discs the pupils could put a mark on the shape on their sheet to indicate that it has been 'called'. If discs are used the same sheets can be used over again. The teacher may find it useful to make a master sheet and duplicate a sufficient number for the class. Some of the shapes which are held up should be similar but not identical to those on the player's sheets.

This activity can be adjusted for older pupils by increasing the similarities and complexity of the shapes on the sheets.

#### ACTIVITY 10

Purpose: To create an awareness of lines and angles and to provide experience in forming lines and angles with some degree of accuracy.

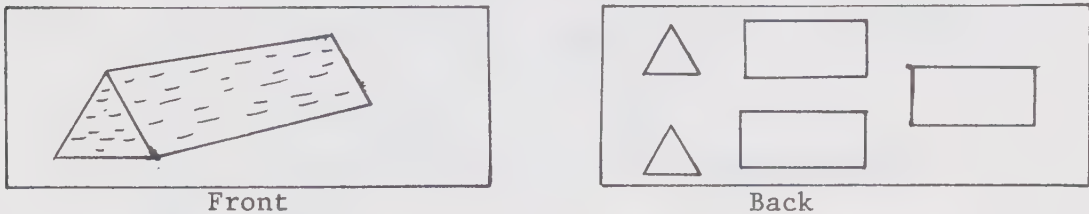
Materials: Geoboards, supply of rubber bands and a number of geometric solids. Cards containing pictures of geometric solids and their faces.

Procedure: The activity requires pupils to trace the faces of the various shapes by using geoboards and rubber bands. Some pupils might like to trace the face of the solids on paper and then make the shape on a geoboard. The procedure which is adopted should



be the one most appropriate for the individual pupil.

As a variation in this activity, cards containing pictures of geometric solids could be used. The object of this activity is similar to the previous one except pupils do not have the actual objects. The back of the cards contain pictures of the faces of each object. Pupils can check their work by turning the cards. An example of a card is shown below.



Advanced pupils might be given the additional activity of naming the shapes of the faces or counting the number of sides and corners on each face.

### ACTIVITY 11

Purpose: To provide experience in constructing triangles and squares and in comparing those shapes.

Materials: Supply of square sheets of paper (approximately 8" by 8"), scissors, glue and large sheets of manilla paper. Assignment cards containing instructions.

Procedure: Each pupil or group receives an instruction card and the necessary materials. (For very young children it will be necessary for the teacher to read the instructions.)

The first card might look like this



Take a sheet of colored paper. Cut this sheet of paper to make 4 pieces which are the same shape and size. Be sure there is no paper left over. Glue each piece on your large sheet of paper.

Can you cut another sheet of colored paper into 3 pieces of same shape and size? What is the name of each piece?

Glue those pieces on the large sheet of paper.

Another card might require pupils to cut out 4 triangles and put them together to make a square, which is glued on the sheet of large paper.

Shapes other than triangles and squares can be investigated in this way. The child's involvement by shaping and comparing gives him an increased awareness of the characteristics and relative properties of the shapes involved.

## ACTIVITY 12

Purpose: To provide experiences in constructing two-dimensional shapes involving angles - a Euclidean concept.

Materials: Supply of rubber bands and dowels (or straws and pipe cleaners) and instruction cards.

Procedure: Many children will want to work independently; others may want to work with a partner. Each pupil should have a



supply of the materials which he needs. A period of free activity when children can construct any shapes they wish should precede any planned exercise.

Cards such as the one shown below should accompany the activity for pupils who are able to read. The teacher will find it necessary to guide younger pupils.

Use 4 sticks to make a closed shape on your desk. How many different closed shapes can you make if you use 4 sticks?

Can you make more than 1 closed shape if you use 3 sticks?

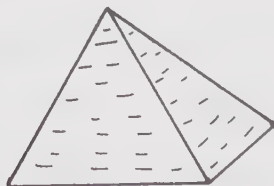
Can you make an open shape using 3 sticks?

Make an open shape using 2 sticks.

Make a box that will stand up by itself.

How many sticks did you use?

Another card might ask the question "Can you make this shape?"



Or a card might ask "Can you make a shape like your classroom?"

"Can you make a shape like your desk?"

### ACTIVITY 13

Purpose: To provide experience in anticipating shapes and





patterns - an integration of Euclidean and projective notions.

Materials: Supply of three dimensional cardboard shapes which can be easily dismantled. Each pupil needs a pencil and a supply of paper.

Procedure: Teacher shows the class a cube and asks them to draw a picture of how the cube would look if it were 'flattened out'. (Teacher demonstrates what she means by 'flattened out'.) Each pupil receives a paper or cardboard cube which they can 'flattened out' and compare with their drawing.

The same procedure might be followed by using a cone or some other shape.

Children sometimes find it an interesting exercise to make a pattern and then fold it to make a shape. This often results in children being quite amused by the unexpected shape and sizes which they form.

#### ACTIVITY 14

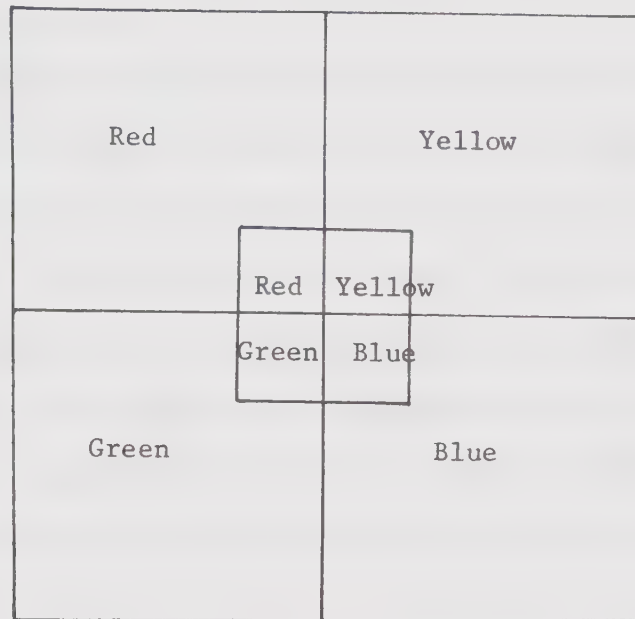
Purpose: To create an awareness of location within the Euclidean plane.

Materials: Four sheets of paper, approximately 8" by 8", each of a different color. (Suppose colors are red, blue, green and yellow.) Four smaller squares of paper, approximately 4" by 4", of the same colors. Chalk or two long strings. An envelope at least 8" by 8".

Procedure: The four small colored squares of paper are glued on one side of a cardboard square 8" by 8" to form a four-colored square to correspond to the colors of the four larger sheets



of paper. The floor of the classroom or the instructional area is divided into quadrants by a chalkline on the floor or by two lengths of string. In each quadrant is placed one of the large colored sheets of paper. One person sits at the intersection of the lines holding the colored card which has been made from the four colored squares. The quadrants of the card are thus colored to match the colored sheets of paper in the quadrants on the floor. The person sitting at the intersection will be called the 'flipper'. An example of the layout is shown below.



The class divides itself into four parts; one quarter of the class to each quadrant (each color). The 'flipper' positions the four-colored square so that the colors correspond to the colors in each quadrant. He then places the card in the envelope without rotating or otherwise turning the card. He then indicates to the first pupil in each quadrant, by turning the envelope, how the colored



card will be 'flipped'. The following are some of the flips which might be performed:

- red goes to green by  $180^\circ$  out-of-plane rotation or  $90^\circ$  counterclockwise in-plane rotation
- red goes to yellow by  $180^\circ$  out-of-plane rotation or  $90^\circ$  clockwise in-plane rotation
- red goes to blue by  $180^\circ$  in-plane rotation
- red goes to red by  $360^\circ$  in-plane rotation.

Obviously there are many possible 'flips' and hence many variations of moves which can be introduced into the activity. Diagonal flips are not permitted.

After the 'flipper' has indicated the 'flip' which he will perform each of the four pupils goes to the color which he thinks is correct. The 'flipper' then removes the card from the envelope and actually 'flips' the card as was indicated. Each pupil then can verify in which quadrant he is supposed to be by looking at the four-colored square. If a pupil is in the wrong quadrant he must return to his original quadrant and wait his turn at the end of the line. The 'flipper' then gives the next four pupils in line a chance to play the game.

Any combination of flips is permissible. After all pupils have moved to other quadrants by the flips of the card, each pupil must indicate to the 'flipper' which flips are necessary to get him back to his original quadrant.

The difficulty of the activity can be increased by using complicated combinations of 'flips'.

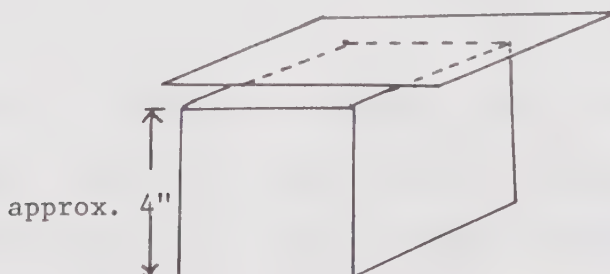


ACTIVITY 15

Purpose: To provide experiences involving rotations in the Euclidean plane.

Materials: Covered milk cartons, paper cut outs of shapes or numerals, glue and plain sheets of paper on which pupils will record their responses.

Procedure: To prepare the milk cartons cut the carton down to about 4". Cover with plain paper. On the inside glue a numeral or picture to each of the four sides. Use a plain piece of cardboard which will completely cover the opening at the top of the box. Pupil's should work in pairs or in small groups. Each pair or group will need a carton and each pupil will need a sheet of paper on which he will record his responses. One pupil chooses a numeral or picture from one of the sides of the carton. The other pupil places the cover on the top and rotates the carton any multiple of  $90^\circ$ , which may be a combination of clockwise and counterclockwise turns. Before the cover is lifted the pupil who picked a numeral or picture indicates where he thinks his choice will be located. The cover is then opened and the pupil verifies his choice and records his answer as being right or wrong. He then takes a turn at rotating the carton as the other pupil makes a choice from the numerals or pictures. Scores are totalled after 10 turns each.





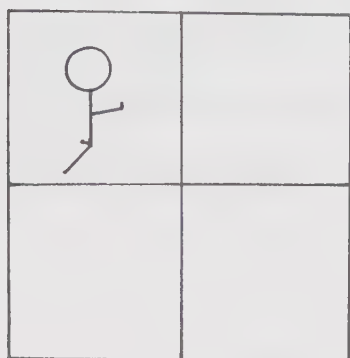


ACTIVITY 16

Purpose: To provide exercise in predicting shapes in reflections - an activity in perspectives.

Materials: A supply of transparencies on which are drawings of various shapes, a supply of small cards containing possible appearances of the shapes on the transparencies after reflection in the Euclidean plane, a sheet of paper the same size as the transparencies and divided into four equal parts by two lines, and a supply of small plain sheets for recording the pupils' responses.

Procedure: Pupils should work in pairs. A transparency containing a shape drawn in one corner is laid on the grid as shown.



Small cards from which pupil may choose.

The small cards containing the appropriate shapes are laid on the desk. They are not permitted to be moved unless the pupil who is predicting what a shape will look like when it is reflected or 'flipped' in a certain way thinks a rotation of a small card is necessary in order to get the correct choice. Placing the small cards prior to flipping is necessary since rotation of the cards constitutes a different transformation. One pupil indicates how the transparency will be 'flipped' and the other pupil makes his



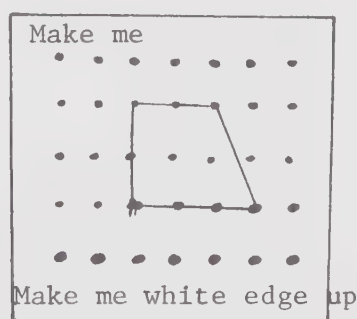
choice from the shapes on the small cards which have been provided. After the transparency has been 'flipped' the pupil can verify his choice and record his answer as being ( $\checkmark$ ) or wrong (x). Pupils then change roles and the activity continues.

### ACTIVITY 17

Purpose: To provide experiences in predicting shapes by rotations - exercise in co-ordination of perspectives.

Materials: Geoboards, geopaper, supply of rubber bands.

Procedure: The edges of the geoboard should be painted or covered with colored tape. For example, suppose the edges are colored, red, blue, green and white. Assignment cards containing instructions should be included for pupils who can read for themselves. The teacher will find it necessary to give oral instructions to younger children. An example of the assignment cards is show below.

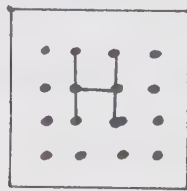


All shapes are made at first with the white edge of the geoboard nearest the pupil. The assignment card indicates which transformation is to be considered in making the shape the second time. The assignment card above asks the pupil to make the shape on his geoboard with the white edge nearest him. Then the pupil is to trace the shape on a sheet of geopaper as it would appear if the board had the white edge turned up, that is, if the geoboard were rotated  $180^\circ$ .



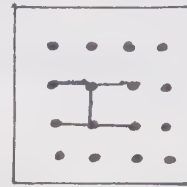
The pupil verifies his response by rotating the geoboard to the appropriate place.

At first shapes which can be easily labelled before and after rotation should be attempted. Young children seem to get satisfaction from being able to name their constructions. An example is shown.



White edge

"H" becomes "T"  
on a line



White  
edge

The rotations can be any multiple of  $90^\circ$  in either a clockwise or counterclockwise direction.

### ACTIVITY 18

Purpose: To assist in developing projective spatial abilities by forming and viewing shadows of objects.

Materials: Light source. Some form of artificial light or sunlight could be used as a light source. The latter is preferred since it does not involve the setting up of any materials and consequently involves less obvious scientific aspects which could distract from the purpose of the activity.

Procedure: Start with pupils thinking of an image they would like to make on the opposite wall of the classroom or on their desk by placing their hands in the path of the sun's ray. Silhouettes of dogs and rabbits are not difficult to make in this way. Some children might like to use their whole body to form a silhouette. All the class can be involved, each taking a turn at forming the par-



ticular image which they have chosen and displaying it for the class to see.

A next step in this kind of activity might involve the different silhouettes which are possible from the objects commonly found in a classroom - pencil, eraser, book, ball, circular disk and so on. Children will usually experiment quite enthusiastically with this activity without much direction from the teacher. Towards the end of the activity the teacher should encourage discussion on the comparison of shapes and their images.

More advanced pupils might like to move in different positions relative to an object which is being held by his partner to try and find a viewpoint which resembles the image being cast by the sun's rays. Two pupils could take turns trying to find the corresponding location for different objects. Many pupils will soon come to a conclusion as to the most appropriate place from which the object should be viewed.

## I. SUMMARY

This chapter contains sample activities for elementary school mathematics based on the investigations into aspects of child development and the spatial abilities of young children.

On the basis of those investigations and the activities included in this chapter it is recommended that activities in the following areas be included in a mathematics program designed for children in elementary grades:

- classification of and familiarity with 3-D objects such as spheres, cubes, cones and polyhedrons





- classification of and familiarity with 2-D objects such as squares, rectangles, triangles and rhombuses and other similar polygons
- topological aspects of closure, proximity, separation and order
- construction of 2-D figures by using pencil-paper, geoboards, paper cut outs and other concrete materials such as plasticine
- construction of 3-D shapes using sticks, straws, plasticine and other materials
- transformations in the Euclidean plane involving reflections and rotations by using geoboards and games related to transformational geometry.



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## APPENDIX A

### TEST OF TOPOLOGICAL SPACE



## TEST OF TOPOLOGICAL SPACE

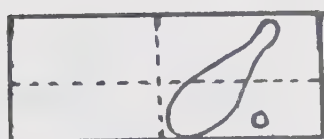
Appropriate items will be scored on the closure, proximity, separation and order properties of topological space. Proximity will be scored by using a 5" by 8" transparent four part grid.

Item  
Number

Shape

Scoring Procedure

1.

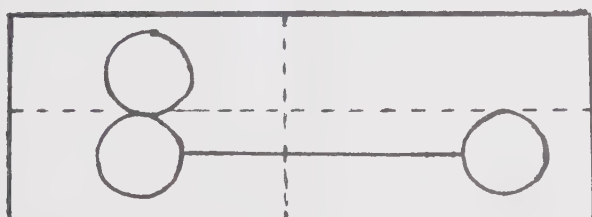


☐ CLOSURE - Each component is a closed figure

☐ SEPARATION - Small circle does not touch larger closed shape

☐ PROXIMITY - A four part grid is placed over the subject's response. The small circle must lie in quadrant four as shown at left.

2.

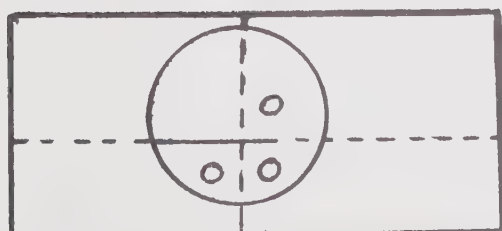
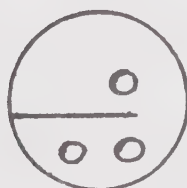


☐ CLOSURE - Each component is a closed figure.

☐ SEPARATION - Small circles do not overlap each other by more than one-tenth the diameter or separated from the end of the straight line by more than one-tenth the diameter.

☐ PROXIMITY- Three circles must lie in the quadrants as shown at left.

3.



☐ CLOSURE - Each component is a closed figure.

☐ SEPARATION - Each small circle is represented inside the larger circle and is disjoint from the larger circle and from each other.

☐ PROXIMITY - Three circles must lie in quadrants as shown at left.



## 4. Linear Order



Beads are presented in circular order. Subject is to place similar beads on stick in the same order.

- ☐ No correspondence in order or if more than three errors in arrangement.
- ☐ If three errors in arrangement
- ☐ If two errors in arrangement
- ☐ If one error in arrangement
- ☐ Accurate representation. No errors in arrangement.

## 5. Linear Order



Beads are presented in the form of a figure eight. Subject is required to place similar beads on stick in the same order.

Scoring as in Model 4



## APPENDIX B

### TEST OF EUCLIDEAN SPACE

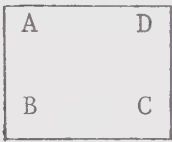
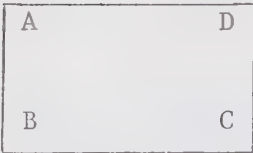
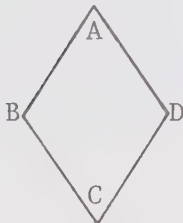




## TEST OF EUCLIDEAN SPACE

Appropriate items will be scored as follows:

1. Closure - To compare with the closure aspect on the test of Topological Space.
2. Presence of angles and straight lines to distinguish topological and euclidean properties.
3. The extent to which attention has been given to relative length of sides and size of angles to investigate proportions and parallels.
4. Abilities with horizontal and vertical axis to investigate subject's abilities to operate in a co-ordinate system.

| Item Number | Shape                                                                                |            | Scoring Procedure                                                                                                                                                          |
|-------------|--------------------------------------------------------------------------------------|------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1.          |     | 1. CLOSURE | 2. PRESENCE OF ANGLES<br>PRESENCE OF STR. LINES<br><br>3. $\angle A = \angle B = \angle C = \angle D + 10^\circ$<br>$AB = BC = CD = AD + \frac{1}{5}$                      |
|             | $AB = BC = CD = AD$ . All Angles = $90^\circ$                                        |            |                                                                                                                                                                            |
| 2.          |   | 1. CLOSURE | 2. PRESENCE OF ANGLES<br>PRESENCE OF STR. LINES<br><br>3. $\angle A = \angle B = \angle C = \angle D + 10^\circ$<br>$AB = DC + \frac{1}{5}$ and<br>$AB = DC + \frac{1}{5}$ |
|             | $AB = CD$ , $AD = BC$ . $AB = \frac{1}{2}BC$ .<br>All Angles = $90^\circ$            |            |                                                                                                                                                                            |
| 3.          |   | 1. CLOSURE | 2. PRESENCE OF ANGLES<br>PRESENCE OF STR. LINES<br><br>3. $\angle A = \angle C + 10^\circ$ and<br>$\angle B = \angle D + 10^\circ$<br>$AB = BC = CD = AD + \frac{1}{5}$    |
|             | $AB = BC = CD = AD$ .<br>Angles A and D = $140^\circ$<br>Angles B and C = $40^\circ$ |            |                                                                                                                                                                            |



4.



$AB=BC$ . Angles =  $60^\circ$

1. CLOSURE

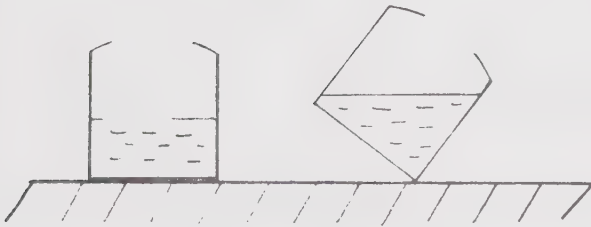
2. PRESENCE OF ANGLES  
PRESENCE OF STR. LINES

3.  $\angle A = \angle B = \angle C + 10^\circ$

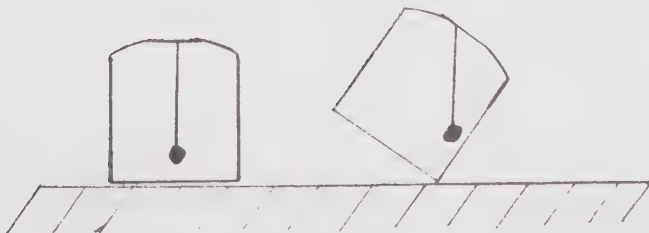
$AB=BC=AC+1/5$

## Description

5. Horizontal Axis Material:  
One narrow-necked bottle with straight parallel sides and a flat base, about  $1/4$  full of a colored liquid. An empty bottle similar to the other bottle. Subjects will be given appropriate pictures of the bottle.



6. Vertical Axis Material:  
A narrow necked bottle with straight parallel sides and a flat base. A string with a small weight attached is fixed so that the string hangs parallel to the sides of the bottle. The subject will be supplied with appropriate pictures of the bottle.



## Scoring Procedure

Draw the picture you see here.  
(Investigator indicates the straight side bottle containing liquid.)

If the bottle with water is used will the water stay where it is or will it move?

Draw on your picture of the bottle (tilted) where the water will be when it is tipped.

(Bottle is tipped for child to see.)  
On another picture draw what you see.

Draw a picture of what you see here. (Investigator indicates the line and weight hanging in the bottle.)

When the bottle is tipped (indicate with an empty bottle) draw how you think the string and weight will look.

(Investigator tips bottle containing line and weight.) Draw what you see here.



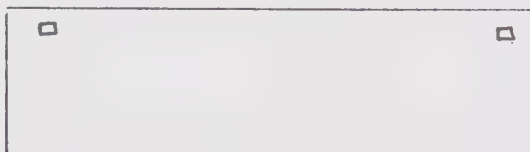
## APPENDIX C

### TEST OF PROJECTIVE SPACE



## TEST OF PROJECTIVE SPACE

PROBLEM 1. Place the rectangular base, two houses and ten matchsticks in clay base within easy reach of the subject. The investigator tells the subject that the new game is to make a straight fence between the two houses. (Investigator places houses in appropriate positions as shown.)



At this point a short training session is necessary in order to ensure that the subject is aware of what a straight line is. A straight line is drawn by the investigator and the subject's attention is directed to this. A number of other straight and curved lines are drawn until it is clear that the subject can distinguish a straight line.

Now go back to the game of putting up a straight fence between the two houses. Emphasize that the line must be very straight. The investigator must not trace the straight line between the two houses with his finger. Be sure that the subject understands the instructions but do not interrupt him if he makes a mistake. Note any corrections subject may make.

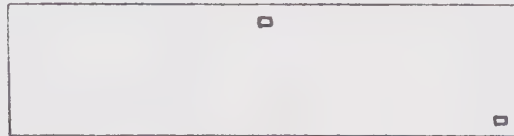
PROBLEM 2. Dismantle the line which the subject has made, put all the matchsticks back together beside the rectangular base. Place the houses as shown. Make certain that the subject understands the instructions. Procedure similar to that in Problem 1.



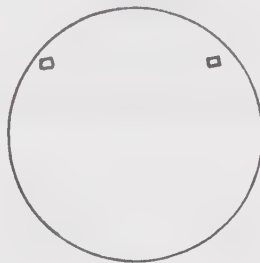




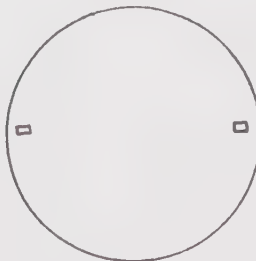
PROBLEM 3. Dismantle the line which the subject has made and reposition houses as shown. Proceed as in Problem 2.



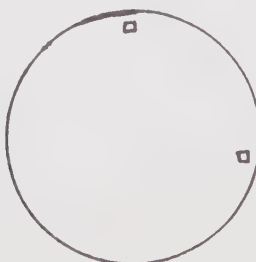
PROBLEM 4. The circular disk is used instead of rectangular disc. Houses are positioned as shown. Proceed as in Problem 2.



PROBLEM 5. Dismantle the line which the subject has made. Reposition houses as shown. Proceed as in Problem 2.



PROBLEM 6. Reposition houses as shown and proceed as in Problem 2.





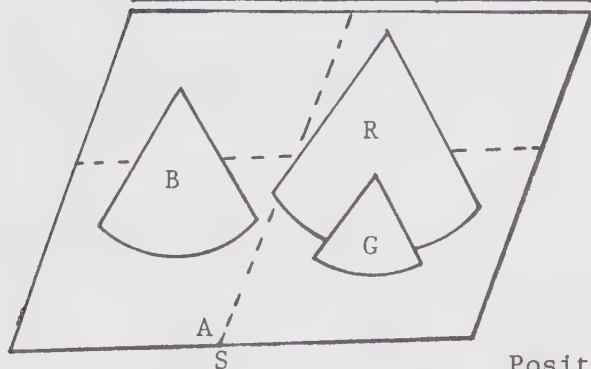
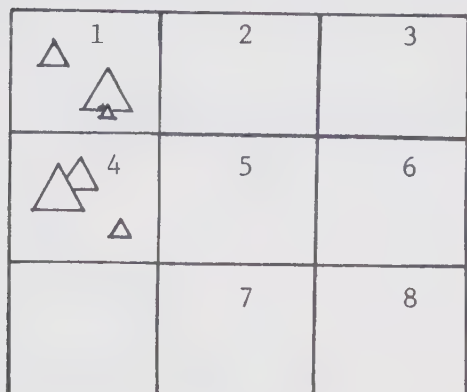
## APPENDIX D

### TEST OF CO-ORDINATION OF PERSPECTIVES

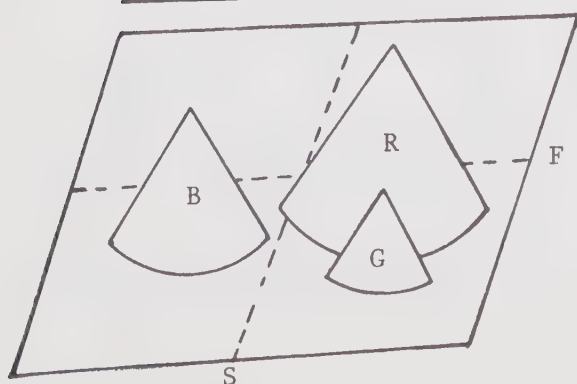
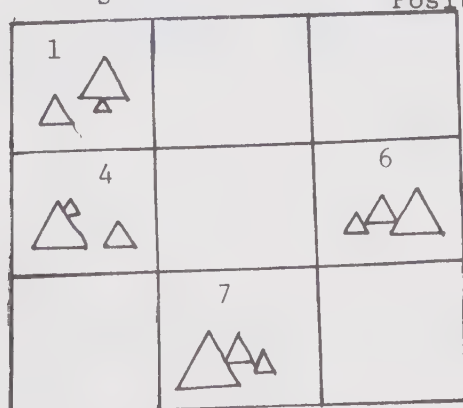


## TEST OF CO-ORDINATION OF PERSPECTIVES

S - indicates position of SUBJECT. A, F, B, E and C indicate position of 'photographer'. R = Red Cone, B = Blue Cone, G = Green Cone

Training Sample

Position 1

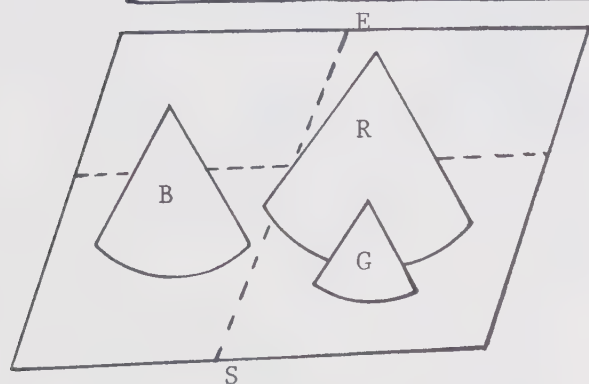
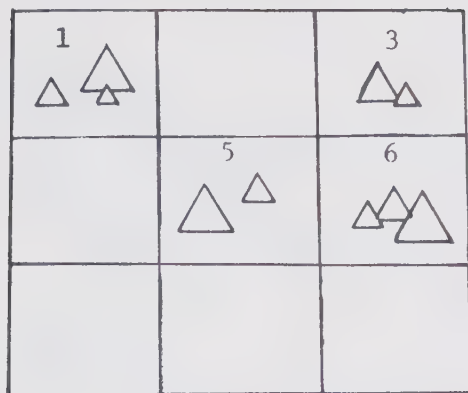
Procedure

Subject is seated in front of display and the model of the 'photographer' is placed as indicated (A). Sketches 1 and 4 are shown as choices of the 'photographer'. The investigator discusses with the subject why 1 is the correct choice and 4 the incorrect choice. This item is NOT scored.

POSITION 1. Cover sketches 1 and 4. Model of 'photographer' is moved to position F. The subject is asked to pretend that the model is taking a picture of the 'mountains' (represented by the cones). The choices 1, 4, 6 and 7 are then exposed and the subject is asked to choose the sketch corresponding to the view of the 'photographer'. Vocabulary and stories are chosen as are considered appropriate for each individual. The subject is encouraged to justify his choice by verbalizing but this does not influence his score on any item.

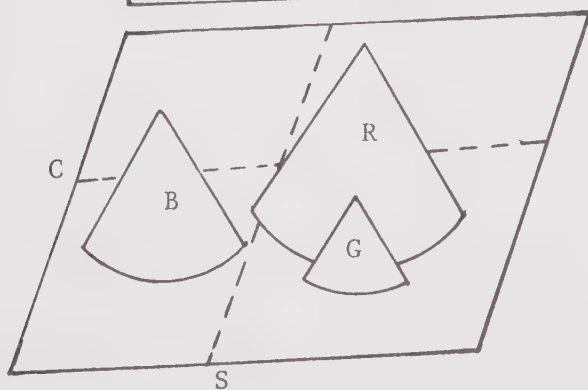
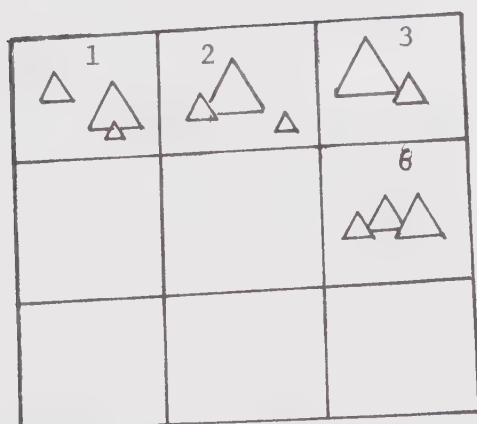


Position 2



POSITION 2. Cover all sketches. Reposition the model at E. Procedure as in Position 1. Expose sketches 1, 3, 5 and 6 as choices.

Position 3

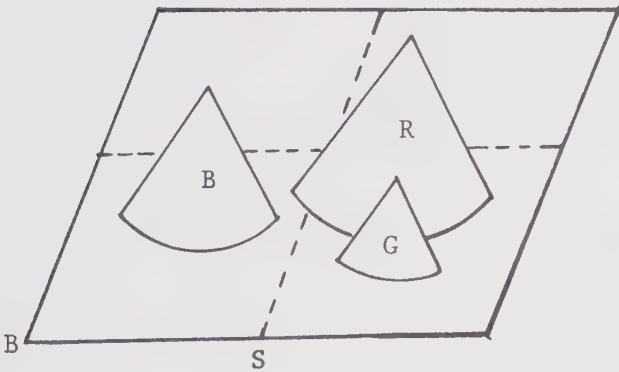
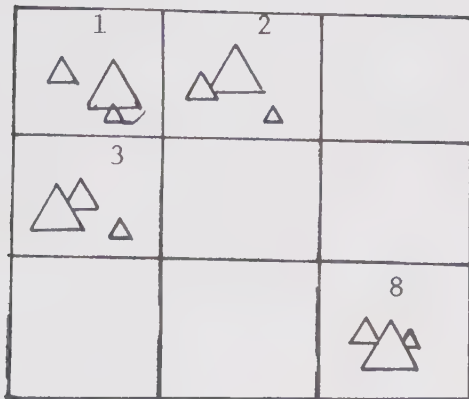


POSITION 3. Cover all sketches. Reposition model to C. Procedure as in Position 1. Expose sketches 1, 2, 3 and 6.





Position 4



POSITION 4. Cover all sketches. Reposition model to B. Procedure as in Position 1. Expose sketches 1, 2, 4 and 8.





















**B30043**